Existence of Three Weak Solutions for Elliptic Dirichlet Problem

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Abstract
The aim of this paper is to establish the existence of at least three weak solutions for the elliptic Dirichlet problem. Our main tool is a three critical point theorem and Therorem3.1. of Gabriele Bonanno, Giovanni Molica Bisci [4].

Keywords: Dirichlet problem; Critical points; Three noitulos

1. Introduction
In this paper we investigate the following elliptic Dirichlet problem

\[
\begin{aligned}
-\Delta u &= \lambda f(x,u) - a(x)u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{aligned}
\]  

(1.1)

where \( \Omega \) is a non empty bounded open subset of the Euclidean space \( \mathbb{R}^N \), \( N \geq 3 \), with boundary of class \( C^1 \), \( \lambda \) is a positive parameter and \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a function, and the positive weight function \( a(x) \in C(\Omega) \).

Existence of three solutions for different kinds of Dirichlet problem has been widely studied in literature, see for instance[1, 3, 5, 6, 7].

2. Preliminaries
Our main tool is the following critical point theorem.

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Theorem 2.1. Let $X$ be a reflexive real Banach space, $\phi: X \to R$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$, $\psi: X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\phi(0) = \psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \phi(\bar{x})$, such that:

$$(a_1) \frac{\sup_{\phi(x) \leq \psi(x)} \psi(x)}{r} < \frac{\psi(x)}{\phi(x)};$$

$$(a_2) \text{ hcaer of } \Lambda_r := \left\{ \psi(x) \phi(\bar{x}) - r \sup_{\phi(x) \leq \psi(x)} \psi(x) \right\} \text{ the functional } \phi - \lambda \psi \text{ is coercive.}$$

Then, for each $\lambda \in \Lambda_r$, the functional $f_\lambda := \phi - \lambda \psi$ has at least three distinct critical points in $X$. Here and in the sequel, $f : \Omega \times R \to R$ is a Caratheodory function such that

$$(h_1) \text{ There exist two non negative constants } a_1, a_2 \text{ and } q \in [1, \frac{2N}{(N-2)}] \text{ such that}$$

$$|f(x,t)| \leq a_1 + a_2 |t|^{q-1}, \quad (2.1)$$

for every $(x,t) \in \Omega \times R$.

We recall that the symbol $H^1_0(\Omega)$ indicates the closure of $C_0^{\infty}(\Omega)$ in the Sobolev space $W^{1,2}(\Omega)$, with respect to the norm

$$||u|| := (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}}$$

and we define

$$||u||_j := \left( \int_\Omega (|\nabla u(x)|^2 + a(x)|u(x)|^2) dx \right)^{\frac{1}{2}}$$

then there exist positive suitable constants $c_1, c_2$ such that

$$c_1 ||u|| \leq ||u||_j \leq c_2 ||u||$$

and a function $u: \Omega \to R$ is said to be a weak solution of (1.1) if $u \in H^1_0(\Omega)$ and

$$\int_\Omega \nabla u(x) \nabla v(x) dx - \lambda \int_\Omega f(x, u(x)) v(x) dx = - \int_\Omega a(x) u(x) v(x) dx$$

for all $v \in H^1_0(\Omega)$.

In order to study problem (1.1), we will use the functionals $\phi, \psi: H^1_0(\Omega) \to R$ defined by putting

$$\phi(u) := \frac{||u||^2}{2},$$

and

$$\psi(u) := \int_\Omega F(x, u(x)) dx, \quad \forall u \in H^1_0(\Omega),$$

where

$$F(x, \xi) := \int_0^\xi f(x, t) dt,$$

for every $(x, \xi) \in \Omega \times R$.

From [4] clearly $\phi: X \to R$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$. On the other hand, $\psi$ is well defined, continuously Gâteaux differentiable and with compact derivative. More precisely, one has

$$\phi(u)(v) = \int_\Omega (\nabla u(x), \nabla v(x) + a(x) u(x) v(x)) dx,$$

$$\psi(u)(v) = \int_\Omega f(x, u(x)) v(x) dx,$$
for every \( u, v \in H^1_0(\Omega) \).

A critical point of the functional \( J_\lambda := \phi - \lambda \psi \) is a function \( u \in H^1_0(\Omega) \) such that
\[
\phi(u)(v) - \lambda \psi(u)(v) = 0, \quad (2.2)
\]
for every \( v \in H^1_0(\Omega) \). Hence the critical points of the functional \( J_\lambda \) are weak solutions of problem (1.1). Now, put \( 2^* = \frac{2N}{(N-2)} \) and denote, as usual, with \( \Gamma \) the Gamma function defined by
\[
\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} dz, \quad \forall t > 0.
\]
From the Sobolev embedding theorem there exist \( c \in R^+ \) such that
\[
||u||_{L^{2^*}(\Omega)} \leq c ||u||, \quad u \in H^1_0(\Omega). \quad (2.3)
\]
The best constant that appears in (2.3) is
\[
c = \frac{1}{\sqrt{N(N-2)} \pi} \left( \frac{N!}{2^t \Gamma(\frac{N}{2})} \right)^{\frac{1}{N}}, \quad (2.4)
\]
Fixing \( q \in [1, 2^*] \), again from the Sobolev embedding theorem, there exists a positive constant \( c_q \) such that
\[
||u||_{L^q(\Omega)} \leq c_q ||u||, \quad u \in H^1_0(\Omega), \quad (2.5)
\]
and, in particular, the embedding \( H^1_0(\Omega) \hookrightarrow L^q(\Omega) \) is compact.

Due to (2.4), as simple consequence of Holder’s inequality, it follows that
\[
c_q \leq \frac{\text{meas}(\Omega)^{\frac{2^* - q}{2}}}{{\sqrt{N(N-2)} \pi}} \left( \frac{N!}{2^t \Gamma(\frac{N}{2})} \right)^{\frac{1}{N}}, \quad (2.6)
\]
where \( \text{meas}(\Omega) \) denotes the Lebesgue measure of the set \( \Omega \).

Moreover, let
\[
D := \sup_{x \in \Omega} \text{dist}(x, \partial \Omega). \quad (2.7)
\]
Simple calculations show that there is \( x_0 \in \Omega \) such that \( B(x_0, D) \subseteq \Omega \).

Finally, we set
\[
k := \frac{D^{\frac{N}{2}}}{2\pi} \left( \frac{\Gamma(1 + \frac{N}{2}) \sqrt{\pi}}{D^{N - \frac{N}{2}}} \right)^{\frac{1}{2}}, \quad (2.8)
\]
and
\[
K_1 := \frac{2\sqrt{2} c_1 (2^N - 1)}{D^2}, \quad K_2 := \frac{2^q \gamma^2 c_q^q (2^N - 1)}{qD^2}. \quad (2.9)
\]

3. Conclusion

Our main result is the following theorem.

\textbf{Lemma 3.1.} Let \( f: \Omega \times R \rightarrow R \) be a Caratheodory function such that \((h_1)\) holds. Assume that
\( (h_2) F(x, \xi) \geq 0 \) for every \( (x, \xi) \in \Omega \times R^+; \)
\( (h_3) \) there exist two positive constants \( b \) and \( s < 2 \) such that
\[
F(x, \xi) \leq b(1 + |\xi|^s),
\]
for almost every \( x \in \Omega \) and for every \( \xi \in R; \)
\( (h_4) \) there exist two positive constants \( \gamma \) and \( \delta \), with \( \delta > \gamma k \) such that
\[
\frac{\inf_{x \in \Omega} F(x, \delta)}{E \delta^2} \leq a_1 \frac{K_1}{\gamma} + a_2 K_2 \gamma^{q - 2},
\]
where \( a_1, a_2 \) are given in \((h_1)\) and \( k, K_1, K_2 \) are given by (2.8) and (2.9).

Then, for each parameter \( \lambda \) belonging to...
the problem (1.1) possesses at least three weak solutions in $H^1_0(\Omega)$. 
Proof: Let us apply theorem 2.1 with $X=H^1_0(\Omega)$ and 
\[ \phi(u) := \frac{1}{2} |u|_2^2, \quad \psi(u) := \int_\Omega F(x,u(x))dx, \]
for every $u \in X$. Let $\lambda > 0$ and put 
\[ j_\lambda(u) := \phi(u) - \lambda \psi(u), \quad \forall u \in X. \]
As observed in section 2, $\phi : X \to \mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $X^*$. Moreover, $\psi$ is continuously Gâteaux differentiable with compact derivative and $\phi(0) = \psi(0) = 0$. 

Owing to $(h_1)$, one has that 
\[ F(x,\xi) \leq a_1 |\xi| + a_2 \frac{|\xi|^q}{q}, \quad (3.1) \]
for every $(x,\xi) \in \Omega \times \mathbb{R}$. 
Let $r \in ]0, +\infty[$ and consider the function 
\[ \chi(r) := \frac{\sup_{u \in \phi^{-1}([0, r])} \psi(u)}{r}. \]
Taking into account (3.1) it follows that 
\[ \psi(u) = \int_\Omega F(x,u(x))dx \leq a_1 ||u||_{L^1(\Omega)} + \frac{a_2}{q} ||u||_{L^q(\Omega)}^q. \]
Then, for every $u \in X: \phi(u) \leq r$, due to (2.5), we get 
\[ \psi(u) \leq \left( \sqrt{2r} c_1 a_1 + \frac{2^{q-1} c_2 a_2}{q} r^{\frac{q}{q-1}} \right). \]
Hence 
\[ \sup_{u \in \phi^{-1}([0, r])} \psi(u) \leq \left( \sqrt{2r} c_1 a_1 + \frac{2^{q-1} c_2 a_2}{q} r^{\frac{q}{q-1}} \right). \]
Since, from (3.2), the following inequality holds 
\[ \chi(r) \leq \left( \sqrt{\frac{2}{r}} c_1 a_1 + \frac{2^{q-1} c_2 a_2}{q} r^{\frac{q}{q-1}} \right), \]
for every $r > 0$. 
Next, put 
\[ u_\delta(x) := \begin{cases} 
0 & \text{if } x \in \Omega - B(x_0,D), \\
\frac{2\delta}{D} (D - |x - x_0|) & \text{if } x \in B(x_0,D) - B \left( x_0, \frac{D}{2} \right), \\
\delta & \text{if } x \in B \left( x_0, \frac{D}{2} \right). 
\end{cases} \]
Clearly $u_\delta \in X$ and we have 
\[ \phi(u_\delta) = \frac{1}{2} \left( \int_\Omega (|\nabla u_\delta(x)|^2 + a(x)|u_\delta(x)|^2)dx \right) \]
\[
\int_{B(x_0, \delta)} a(x) \frac{(2\delta)^2}{D^2} \, dx + \int_{B(x_0, \delta)} a(x) \frac{(2\delta)^2}{D^2} \, |D - |x - x_0||^2 \, dx + \\
\int_{B(x_0, \delta)} a(x) \delta^2 \, dx \\
\leq \frac{1}{2} \left( \frac{(2\delta)^2}{D^2} \frac{\pi^N}{\Gamma \left( \frac{1 + N}{2} \right)} \left( D^N - \left( \frac{D}{2} \right)^N \right) \right)
\]

\[
+ \frac{(2\delta)^2}{D^2} \sup_{x \in \Omega} a(x) \max_{x \in \partial B(x_0, \delta)} |D - |x - x_0||^2 \frac{\pi^N}{\Gamma \left( \frac{1 + N}{2} \right)} \left( D^N - \left( \frac{D}{2} \right)^N \right) \\
+ \delta^2 \sup_{x \in \Omega} a(x) \frac{\pi^N}{\Gamma \left( \frac{1 + N}{2} \right)} \left( D^N - \left( \frac{D}{2} \right)^N \right)
\]

By assumption (h2), we infer

\[
\psi(u_\delta) = \int_{\Omega} F(x, u_\delta(x)) \, dx \geq \int_{B(x_0, \delta)} F(x, \delta) \, dx \\
\inf_{x \in \Omega} F(x, \delta) \frac{\pi^N}{\Gamma \left( \frac{1 + N}{2} \right)} \left( D^N - \left( \frac{D}{2} \right)^N \right).
\]

Hence, by (3.5) and (3.6) one has

\[
\frac{\psi(u_\delta)}{\phi(u_\delta)} \geq \frac{D^2}{2(2N-1)} \frac{\inf_{x \in \partial B(x_0, \delta)} F(x, \delta)}{\delta^2 \cdot E}.
\]

In view of (3.3) and taking into account (h4), we get

\[
\chi(y^2) = \frac{\sup_{u \in \mathcal{U} \cap \partial \Omega} \psi(u)}{y^2} \leq \left( \sqrt{2} \frac{c_1}{\gamma} + \frac{2q_c q_a}{\gamma^2} \right) y^{q-2}
\]

\[
= \frac{D^2}{2(2N-1)} \left( a_1 + \frac{a_2 K_2}{\gamma} + \frac{a_2 K_2}{\gamma} \right) y^{q-2} < \frac{D^2}{2(2N-1)} \frac{\inf_{x \in \partial B(x_0, \delta)} F(x, \delta)}{\delta^2 \cdot E} \leq \frac{\psi(u_\delta)}{\phi(u_\delta)}.
\]

Therefore, the assumption (a1) of theorem 2.1 is satisfied.

Moreover, if \( s < 2 \), for every \( u \in X \), \( |u|^s \in L^2(\Omega) \) and the H"older's inequality gives

\[
\int_{\Omega} |u(x)|^s \, dx \leq ||u||^s_{L^2(\Omega)} \text{meas}(\Omega)^{\frac{2-s}{2}}, \quad \forall u \in X.
\]

Then, by (2.5), one has

\[
\int_{\Omega} |u(x)|^s \, dx \leq c_2^s ||u||^s \text{meas}(\Omega)^{\frac{2-s}{2}}, \quad \forall u \in X.
\]

From (3.8) and due to condition (h3), it follows that

\[
J_4(u) \geq \frac{||u||^2}{2} - \lambda b \text{meas}(\Omega)^{\frac{2-s}{2}} ||u||^s - \lambda b \text{meas}(\Omega), \quad \forall u \in X.
\]

Therefore, \( J_4 \) is a coercive functional for every positive parameter, in particular, for every
\[ \lambda \in \Lambda(\gamma, \delta) \subseteq \frac{\phi(u_\delta)}{\psi(u_\delta)}, \frac{\gamma^2}{\sup_{\psi(u) \leq \gamma^2}}. \]

Then, also condition \((a_2)\) holds. Hence, all the assumptions of theorem 2.1 are satisfied, so that, for each \(\lambda \in \Lambda(\gamma, \delta)\) the functional \(J_\lambda\) has at least three distinct critical points that are weak solutions of the problem (1.1).

**Example 3.1** Let \(\Omega\) be an open ball of radius one in \(\mathbb{R}^4\), \(q:=3\in]2,4[\) and \(s:=\frac{3}{2} < 2\).

Pick \(r:=200\) and consider the function \(f : \mathbb{R} \rightarrow \mathbb{R}\) defined by

\[
 f(t):= \begin{cases} 
 1 + t^2 & \text{if } t \leq 200, \\
 1 + 2000 \sqrt{2}t & \text{if } t > 200. 
\end{cases} 
\]

and \(a(x) = \frac{1}{e^{x^2}}\).

Then, by theorem 3.1, for each

\[ \lambda \in \left[ \frac{18000E}{40003}, \frac{12\pi^{1/4}}{1 + 2\sqrt{3}\pi^2} \right] 4\pi, \]

the problem (1.1) possesses at least three weak positive solutions in \(H^1_0(\Omega)\).

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**References**