Duality and biorthogonality for g-frames in Hilbert spaces

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Abstract

The main aim of this paper is to define the generalized Riesz-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases. We characterize exactly properties of the first sequence in terms of the associated one, which yields duality relations for the abstract g-frame setting. ©2017 all rights reserved.

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1. Introduction

Duality principles in Gabor theory such as the Ron-Shen duality principle [13] and the Wexler-Raz biorthogonality relations [17] play a fundamental role for analyzing Gabor systems. Casazza et al. in [4] introduced a general approach to derive duality principles in abstract frame theory. For each sequence in a separable Hilbert space they defined a Riesz-dual sequence dependent only on two orthonormal bases. They characterize exactly properties of the first sequence in terms of the Riesz-dual sequence, which yields duality relations for the frame setting. Frames were first introduced by Duffin and Schaeffer [9] in the context of nonharmonic Fourier series and reintroduced in 1986 by Daubechies et al. in [8]. Currently, frames play important roles in many applications in mathematics, science, and engineering such as signal processing, image processing, data compression, etc.

Let \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) be orthonormal bases for a separable Hilbert space \( \mathcal{H} \) and let \( f = \{f_i\}_{i \in I} \) be any sequence in \( \mathcal{H} \) for which \( \sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty \) for all \( j \in I \). Then the Riesz-dual sequence (R-dual sequence) of \( \{f_i\}_{i \in I} \) with respect to \( \{e_i\}_{i \in I} \) and \( \{h_i\}_{i \in I} \) as the sequence \( \{W^f_j\}_{j \in I} \) is given by:

\[
W^f_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \quad \forall j \in I.
\]

This simple construction gives a powerful tool for deriving duality principles in general frame theory. There exists a symmetric relation between the sequences \( \{W^f_j\}_{j \in I} \) and \( \{f_i\}_{i \in I} \) as follows:

\[
f_i = \sum_{j \in I} \langle W^f_j, h_i \rangle e_j, \quad \forall i \in I.
\]

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In particular, this shows that \( \{f_i\}_{i \in I} \) is the R-dual sequence for \( \{\mathcal{H}^f_i\}_{i \in I} \) with respect to \( \{h_i\}_{i \in I} \) and \( \{e_i\}_{i \in I} \). We refer the reader to the articles [6, 7, 14, 18] for an introduction about the theory and applications of R-dual sequences.

Recently, Sun in [15, 16] and Casazza and Kutyniok in [3] introduced a generalization of frames which covers many other recent generalizations of frames, e.g., bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames, and a class of time-frequency localization operators. Sun showed that all of the above applications of frames are special cases of generalized frames.

Let \( \mathcal{K} \) and \( \mathcal{X} \) be two separable Hilbert spaces and let \( \{V_i\}_{i \in I} \) be a family of closed subspaces of \( \mathcal{K} \) and \( B(\mathcal{H}, V_i) \) denote the collection of all bounded linear operators from \( \mathcal{H} \) into \( V_i \) for all \( i \in I \). Then, \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\} \) is a generalized frame or simply a g-frame for \( \mathcal{K} \) with respect to \( \{V_i\}_{i \in I} \) if there exist constants \( 0 < C \leq D < \infty \) such that:

\[
C\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{K}.
\]

The constants \( C \) and \( D \) are called g-frame bounds. If only the right-hand inequality of (1.1) is required, we call it a g-Bessel sequence. Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following examples.

**Example 1.1.** Let \( \mathcal{K} = \mathbb{C}^n \) and \( V_1 = V_2 = \ldots = V_n = C^{n+1} \). Define

\[
\Lambda_1 = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}, \ldots, \quad \Lambda_n = \begin{bmatrix}
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}.
\]

Then, the set \( \Lambda = \{\Lambda_i\}_{i=1}^n \) is a g-frame for \( \mathbb{C}^n \) with respect to \( C^{n+1} \) with g-frame bounds \( \Lambda = 2 \) and \( B = n + 1 \). To see this explicitly, note that for any \( f = (z_1, z_2, \ldots, z_n) \) in \( \mathbb{C}^n \), we have

\[
\sum_{i=1}^n \|\Lambda_i f\|^2 = 2|z_1|^2 + 3|z_2|^2 + \ldots + (n+1)|z_n|^2.
\]

From this, we have

\[
2\|f\|^2 \leq \sum_{i=1}^n \|\Lambda_i f\|^2 \leq (n+1)\|f\|^2.
\]

In frames theory an input signal is represented by a collection of scalar coefficients that measure the projection of that signal onto each frame vector. The representation space employed in this theory equals \( \ell^2(\mathbb{C}) \). However, in g-frames theory an input signal is represented by a collection of vector coefficients that represent the projection (not just the projection energy) onto each subspace. Therefore the representation space employed in this setting is

\[
\left( \sum_{i \in I} \oplus V_i \right)_{\ell^2} = \left\{ \{g_i\}_{i \in I} | g_i \in V_i, \sum_{i \in I} \|g_i\|^2 < \infty \right\}.
\]

In order to analyze a signal \( f \in \mathcal{H} \), i.e., to map it into the representation space, the analysis operator \( T_\Lambda : \mathcal{H} \to \left( \sum_{i \in I} \oplus V_i \right)_{\ell^2} \) given by \( T_\Lambda f = \{\Lambda_i f\}_{i \in I} \) is applied. The associated synthesis operator, which provides a mapping from the representation space to \( \mathcal{H} \), is defined to be the adjoint operator \( T_\Lambda^* : \left( \sum_{i \in I} \oplus V_i \right)_{\ell^2} \to \mathcal{H} \), which is given by \( T_\Lambda^*\{g_i\}_{i \in I} = \sum_{i \in I} \Lambda_i^* g_i \). By composing \( T_\Lambda \) and \( T_\Lambda^* \) we obtain the g-frame operator \( S_\Lambda : \mathcal{H} \to \mathcal{H} \), \( S_\Lambda f = T_\Lambda^* T_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f \), which is a positive, self-adjoint and invertible operator and
C ≤ ∥SA∥ ≤ D. The canonical dual g-frame for \{Λi\}i∈I is defined by \(\hat{Λ}_i\) where \(\Lambda_k = \Lambda_i S^{-1}_A\) which is also a g-frame for \(\mathcal{H}\) with respect to \{V_i\}i∈I with \(\frac{1}{D}\) and \(\frac{1}{C}\) as its lower and upper frame bounds, respectively. Also we have
\[
f = \sum_{i \in I} \Lambda_i^* \hat{Λ}_i f = \sum_{i \in I} \hat{Λ}_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}.
\]
Moreover, \(\{\Lambda_i S^{-1}_A\}i∈I\) is a Parseval g-frame for \(\mathcal{H}\) with respect to \{V_i\}i∈I.

Generalized Riesz-dual sequence or simply g-R-dual sequence is a natural generalization of R-dual sequence which provides a powerful tool in the analysis of duality relations in general g-frame theory. The purpose of this paper is to introduce the concept of Riesz-dual sequence for g-frames. We give characterizations of g-R-dual sequences and prove that g-R-dual sequences share many useful properties with R-dual sequences. In this article, we show that in fact for each sequence of operators we can construct a corresponding sequence of operators with a kind of duality relation between them. This construction is used to prove duality principles in g-frame theory, which can be regarded as general versions of several well-known duality principles for g-frames. We also give a generalized version of Riesz-dual sequences.

The content of this paper is as follows: In the rest of this section we will briefly recall the necessary parts from g-bases, g-orthonormal bases, and g-Riesz bases. For more information we refer to [1, 2, 5, 10, 11]. In Section 2, we define the g-R-dual sequence from a g-Bessel sequence with respect to a pair of g-orthonormal bases as generalization of Riesz-dual sequence. In this section, we characterize to which extent the g-R-dual sequence of a g-Bessel sequence depends on the chosen g-orthonormal bases. In Section 3, first we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent). Finally, Section 4 deals with duality principle for g-frames. In this section we study properties of dual g-frames and canonical dual g-frames.

**Definition 1.2.** A generalized Schauder basis or simply a g-basis for \(\mathcal{H}\) with respect to \{W_i\}i∈I is a family of onto operators \(\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j) \mid j \in I\}\) such that for all \(f \in \mathcal{H}\) there exist unique vectors \(g_j \in W_j, i \in I\) with
\[
f = \sum_{j \in I} \Gamma_j^* g_j.
\]
In this case, there exist unique operators \(\Lambda_j \in B(\mathcal{H}, W_j)\) such that
\[
f = \sum_{j \in I} \Gamma_j^* \Lambda_j f = \sum_{j \in I} \Lambda_j^* \Gamma_j f,
\]
for all \(f \in \mathcal{H}\). Moreover, the sequences \(\{\Gamma_j\}i∈I\) and \(\{\Lambda_j\}i∈I\) are g-biorthogonal, i.e., \(\Lambda_j^* \Gamma_j = \delta_{ij} g_i\) for all \(i, j \in I, g_j \in W_j\) and \(\{\Lambda_j\}i∈I\) itself forms a g-basis for \(\mathcal{H}\) with respect to \{W_i\}i∈I that so-called dual g-basis of \(\{\Gamma_j\}i∈I\). A g-basis is an unconditional g-basis, if the series in (1.2) converges unconditionally. Consequently, for a g-basis the ordering in (1.2) can be crucial. If \(\{\Lambda_i\}i∈I\) is a g-basis only for its closed linear span, we call it a g-basic sequence with respect to \{W_i\}i∈I.

**Definition 1.3.** Let \(\{\Xi_i \in B(\mathcal{H}, W_i) \mid i \in I\}\) be a sequence of operators. Then
(i) \(\{\Xi_i\}i∈I\) is a g-complete set for \(\mathcal{H}\) with respect to \{W_i\}i∈I, if \(\mathcal{H} = \text{span}(\Xi_i^* W_i)\) i∈I.
(ii) \(\{\Xi_i\}i∈I\) is a g-orthonormal system for \(\mathcal{H}\) with respect to \{W_i\}i∈I, if \(\Xi_i^* \Xi_j = \delta_{ij} W_i\) for all \(i, j \in I\).
(iii) A g-complete and g-orthonormal system \(\{\Xi_i\}i∈I\) is called a g-orthonormal basis for \(\mathcal{H}\) with respect to \{W_i\}i∈I.

**Definition 1.4.** A sequence \(\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j) \mid j \in I\}\) is called a g-Riesz basis for \(\mathcal{H}\) with respect to \{W_i\}i∈I, if \(\{\Gamma_j\}i∈I\) is a g-complete set for \(\mathcal{H}\) with respect to \{W_j\}j∈J and there exist constants \(0 < A \leq B < \infty\) such that
\[
A \sum_{j \in J} ||g_j||^2 \leq \sum_{j \in J} \Gamma_j^* g_j ||^2 \leq B \sum_{j \in J} ||g_j||^2,
\]
(1.3)
for all sequences \( \{g_j\}_{j \in I} \in \left( \sum_{j \in I} \oplus W_j \right)_\ell. \) We define the g-Riesz basis bounds for \( \{f_j\}_{j \in I} \) to be the largest number \( A \) and the smallest number \( B \) such that this inequality (1.3) holds. If \( \{f_j\}_{j \in I} \) is a g-Riesz basis only for \( \text{span}(\{f_j\}_{j \in I}) \), we call it a g-Riesz basic sequence for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \).

The following well-known characterization of g-orthonormal bases is sometimes more useful which is taken from [2].

**Lemma 1.5.** Let \( \Xi = \{\xi_i\}_{i \in I} \) be a g-orthonormal system for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \). Then the following conditions are equivalent:

(i) \( \Xi \) is a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \).
(ii) \( \sum_{i \in I} \Xi_i \Xi_i = I_\ell \).
(iii) \( \|f\| = \sum_{i \in I} \|\Xi_i f\|^2 \), \( \forall f \in \mathcal{H} \).
(iv) \( \|f\| = \sum_{i \in I} \|\Xi_i f\|^2 \), \( \forall f \in \mathcal{H} \).
(v) \( \forall f, g \geq \sum_{i \in I} \langle \Xi_i f, \Xi_i g \rangle \), \( \forall f, g \in \mathcal{H} \).
(vi) If \( \Xi_i f = 0 \) for all \( i \in I \), then \( f = 0 \).

For any given g-frame there is a natural procedure to construct a g-Riesz basis with the same g-frame bounds, see, e.g., [1] for a proof of this standard result.

**Lemma 1.6.** Let \( \{\xi_j\}_{j \in I} \) be a g-orthonormal system for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \) and \( U : \mathcal{H} \to \mathcal{H} \) a bounded bijective operator. Then the following items hold.

(i) The sequence \( \{\xi_i U^*\}_{i \in I} \) is a g-Riesz basis for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) with g-frame operator \( UU^* \) and optimal bounds \( \frac{1}{\|U\|^2}, \|U\|^2 \).
(ii) The dual g-Riesz basis of \( \{\xi_i U^*\}_{i \in I} \) is \( \{\xi_i U^{-1}\}_{i \in I} \) with g-frame operator \( (UU^*)^{-1} \) and the optimal bounds are \( \frac{1}{\|U\|^2}, \|U^{-1}\|^2 \).
(iii) Let \( \Gamma = \{f_j\}_{j \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \) with optimal bounds \( A, B \). Then \( \{\xi_i S_{\Gamma}^{-1}\}_{i \in I} \) is a g-Riesz basis for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \) with optimal bounds \( A, B \). The dual g-Riesz basis of \( \{\xi_i S_{\Gamma}^{-1}\}_{i \in I} \) is \( \{\xi_i S_{\Gamma}^{-1}\}_{i \in I} \) with optimal bounds \( \frac{1}{B}, \frac{1}{A} \).
(iv) Let \( \Gamma = \{f_j\}_{j \in I} \) be a g-Riesz basis for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \), then \( \{\xi_i S_{\Gamma}^{-1}\}_{i \in I} \) is a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \).
(v) Let \( \Gamma = \{f_j \in B(\mathcal{H}, W_j) \mid j \in I \} \) be arbitrary sequence. If \( \text{span}(\{f_j \}_{j \in I}) = \mathcal{H} \) and

\[
\sum_{j \in I} \|g_j\|^2 = \sum_{j \in I} \|f_j\|^2,
\]

then \( \Gamma = \{f_j\}_{j \in I} \) is a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \).

Let \( \Xi = \{\xi_i\}_{i \in I} \) be a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \). If \( f = \sum_{i \in I} \xi_i g_i \), then the coordinate representation of \( f \in \mathcal{H} \) relative to the g-orthonormal basis \( \Xi \) is \( [f]_\Xi = \{g_i\}_{i \in I} \). In this case \( \{g_i\}_{i \in I} \in \left( \sum_{i \in I} \oplus W_i \right)_\ell \) and \( \|f\| = \|\xi f\|_\ell \).

**Definition 1.7.** Let \( \Xi = \{\xi_i\}_{i \in I} \) and \( \Xi' = \{\xi'_i\}_{i \in I} \) be g-orthonormal bases for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) and \( \{V_j\}_{j \in I} \), respectively. The transition matrix from \( \Xi \) to \( \Xi' \) is the matrix \( B = [B_{ij}] \) whose \( (i,j) \)-entry is \( B_{ij} = \langle \xi_j', \xi_i \rangle \) for all \( i,j \in I \). We also have \( B[f]_\Xi = [f]_{\Xi'} \), where, [f]_\Xi and [f]_{\Xi'} are the coordinate representation of an arbitrary vector \( f \in \mathcal{H} \) in the basis \( \Xi \) and \( \Xi' \), respectively. We show that the transition matrix from \( \Xi' \) to \( \Xi \) is \( B^{-1} = B^* \). Let \( B^* = [B^*_{ij}] \), then \( B^*_{ij} = \langle \xi_j, \xi'_i \rangle = \langle \xi'_i, \xi_j \rangle \) for all \( i,j \in I \). By Lemma 1.5 we have

\[
[BB^*]_{ij} = \sum_{k \in I} B_{ik} B^*_{kj} = \sum_{k \in I} E_k E_k^* E_k^* = E_k \left( \sum_{k \in I} E_k^* E_k \right) E_k = E_k I_{2\ell} E_k^* = E_k^* E_k^* = \delta_{ij} \text{I}_{W_j}.
\]

Similarly, \( [B^* B]_{ij} = \delta_{ij} \text{I}_{W_j} \). This implies that \( BB^* = B^* B = I \), where \( I \) is the identity matrix.

Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following example.
Example 1.8. Let $\mathcal{H} = C^{2n}$ and $W_1 = W_2 = \ldots = W_n = C^2$. Define

$$\Xi_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Xi_n = \begin{bmatrix} 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$ 

A direct calculation shows that $\|\Xi_k\| = 1$ and $\Xi_k \Xi^*_k = \delta_{kk}$ for any $1 \leq k, \ell \leq n$. We also have

$$\sum_{k=1}^{n} \|\Xi_k f\|^2 = \sum_{k=1}^{n} (|z_{2k-1}|^2 + |z_{2k}|^2) = \|f\|^2, \quad \forall f = (z_i)_{i=1}^{2n} \in C^{2n}.$$ 

Therefore $\Xi = \{\Xi_k\}_{k=1}^{n}$ is a g-orthonormal basis for $C^{2n}$ with respect to $C^2$. Similarly, the sequence $\Psi = \{\Psi_k\}_{k=1}^{n}$ defined by

$$\Psi_1 = \begin{bmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Psi_n = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix},$$

is also a g-orthonormal basis for $C^{2n}$ with respect to $C^2$ and the matrix

$$B = [\Psi_i \Xi_j^*]_{n \times n} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the transition matrix from $\Xi$ to $\Psi$. Hence, for any $f \in C^{2n}$ we have $B[f]_\Xi = [f]_\Psi$.

Example 1.9. Let $\mathcal{H} = C^{2n}$ and $W_1 = W_2 = \ldots = W_{2n} = C^2$. Define

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 2 & \ldots & 0 & 0 \end{bmatrix}, \ldots, \Gamma_n = \begin{bmatrix} 0 & 0 & \ldots & 2n-1 & 0 \\ 0 & 0 & \ldots & 0 & 2n \end{bmatrix}.$$ 

Since, for every $g_i = (z_{2i-1}, z_{2i}) \in C^2$, we have $\| \sum_{i=1}^{n} \Gamma_i^* g_i \|^2 = \sum_{i=1}^{2n} i^2 |z_i|^2$. Thus $\{\Gamma_i\}_{i=1}^{n}$ is a g-Riesz basis for $C^{2n}$ with respect to $C^2$ with g-Riesz bounds $1$ and $4n^2$. Moreover, we can write $\{\Gamma_i\}_{i=1}^{n} = \{\Xi_i U \}^*_{i=1}$, where $U$ is a bounded bijective operator defined by

$$U = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2n \end{bmatrix},$$

and $\Xi = \{\Xi_k\}_{k=1}^{n}$ is the g-orthonormal basis defined in Example 1.8.

2. The g-R-dual sequence

In this section we define the g-R-dual sequence from a sequence of operators. Then we exactly characterize to which extent the g-R-dual sequence of a g-Bessel sequence depends on the chosen g-orthonormal bases.

Definition 2.1. Let $\Xi = \{\Xi_i\}_{i \in I}$ and $\Psi = \{\Psi_i\}_{i \in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$, respectively. Let $\Lambda = \{\Lambda_i : \mathcal{H} \to V_i | i \in I\}$ be such that the series $\sum_{i \in I} \Lambda_i^* g_i$ is convergent for all $\{g_i\}_{i \in I} \in (\sum_{i \in I} \oplus V_i)_{\ell^2}$. For all $j \in I$, let

$$\Gamma_j^\Lambda : \mathcal{H} \to W_j, \quad \Gamma_j^\Lambda = \sum_{i \in I} \Xi_i \Lambda_i^* \Psi_i.$$ 

Then $\{\Gamma_i^\Lambda\}_{i \in I}$ is called the generalized Riesz-dual sequence (g-R-dual sequence) for the sequence $\Lambda$ with respect to $(\Xi, \Psi)$.
Notice that the hypothesis that the series $\sum_{i \in I} \lambda^*_i g_i^*$ is convergent for all $\{g_i^\prime\}_{i \in I} \in (\sum_{i \in I} \oplus V_i)^C$ is always fulfilled if the sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ is g-Bessel sequence with respect to $\{V_i\}_{i \in I}$.

**Example 2.2.** Let $\mathcal{H} = C^{2^n}$ and let $\{\Xi_i\}_{i=1}^n$, $\{\Psi_i\}_{i=1}^n$ be the g-orthonormal bases for $\mathcal{H}$ with respect to $C^2$ defined in Example 1.8. Define

$$\Lambda_1 = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}, \ldots, \Lambda_n = \begin{bmatrix} 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$ 

Then, $\Lambda = \{\Lambda_i\}_{i=1}^n$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $C^2$ with g-Bessel bound $B = 3$. The g-R-dual sequence for the sequence $\Lambda$ with respect to $(\Xi, \Psi)$ is defined as follows:

$$\Gamma^\Lambda = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix},$$

which is also a g-Bessel sequence for $\mathcal{H}$ with respect to $C^2$ with g-Bessel bound $B = 3$.

Now, we need an algorithm to invent the process and calculate $\{\Lambda_i\}_{i \in I}$ from the sequence $\{\Gamma^\Lambda_i\}_{i \in I}$.

**Theorem 2.3.** Let $\Xi = \{\Xi_i\}_{i \in I}$ and $\Psi = \{\Psi_i\}_{i \in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$, respectively. Let $\{\Lambda_i\}_{i \in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i \in I}$. Then, for all $i \in I$,

$$\Lambda_i = \sum_{j \in I} \Psi_i (\Gamma^\Lambda_j)^* \Xi_j.$$ 

In particular, this shows that $\{\Lambda_i\}_{i \in I}$ is the g-R-dual sequence for $\{\Gamma^\Lambda_j\}_{j \in I}$ with respect to $(\Psi, \Xi)$.

**Proof.** The definition of $\{\Gamma^\Lambda_j\}_{j \in I}$ implies that for every $i, j \in I$

$$\Psi_i (\Gamma^\Lambda_j)^* = \Psi_i (\sum_{k \in I} \Xi_k \Lambda^*_k \Psi_k)^* = \sum_{k \in I} \Psi_i \Psi^*_k \Lambda_k \Xi^*_j = \sum_{k \in I} \delta_{ik} \Lambda^*_k \Xi^*_j = \Lambda^*_i \Xi^*_j.$$ 

Therefore $\Psi_i (\Gamma^\Lambda_j)^* = \Lambda_i \Xi^*_j$. Now, by Lemma 1.5 we have

$$\Lambda_i = \Lambda_i I_{\mathcal{H}} = \Lambda_i \sum_{j \in I} \Xi^*_j \Xi_j = \sum_{j \in I} \Lambda_i \Xi^*_j \Xi_j = \sum_{j \in I} \Psi_i (\Gamma^\Lambda_j)^* \Xi_j.$$ 

\[\square\]

**Definition 2.4.** Let $\Xi = \{\Xi_j\}_{j \in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ and let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i \in I}$ with the g-frame operator $S_{\Lambda} : \mathcal{H} \rightarrow \mathcal{H}$, respectively. Then the matrix representation of $S_{\Lambda}$ with respect to $\Xi$ is the matrix $[S_{\Lambda}] = [S_{ij}]$, with $S_{ij} = \Xi_i S_{\Lambda} \Xi^*_j$. Therefore

$$[S_{\Lambda}] : (\sum_{i \in I} \oplus W_i)_{\mathcal{H}} \rightarrow (\sum_{i \in I} \oplus W_i)_{\mathcal{H}}, \quad \text{with} \quad [S_{\Lambda} f]_{\Xi} = [S_{\Lambda}] [f]_{\mathcal{H}}, \quad \forall f \in \mathcal{H}.$$ 

Suppose $\Lambda = \{\Lambda_{ij}\}$ with $\Lambda_{ij} = \Lambda_{i} \Xi^*_j$, then $\Lambda^* = [\Lambda^*]_{ij}$ and $\Lambda^*_{ij} = \Xi_i \Lambda^*_j$ for all $i, j \in I$. Therefore

$$\Lambda : (\sum_{i \in I} \oplus W_i)_{\mathcal{H}} \rightarrow (\sum_{i \in I} \oplus V_i)_{\mathcal{H}}, \quad \text{and} \quad \Lambda^* \Lambda : (\sum_{i \in I} \oplus W_i)_{\mathcal{H}} \rightarrow (\sum_{i \in I} \oplus W_i)_{\mathcal{H}}.$$ 

The matrix $\Lambda$ is called the analysis matrix for $\Lambda$ with respect to $\Xi$. A direct calculation shows that for every $f \in \mathcal{H}$ we have $\Lambda[f]_{\Xi} = T_{\Lambda} f$. We also have

$$[\Lambda^* \Lambda]_{ij} = \sum_{k \in I} [\Lambda^*]_{ik} [\Lambda]_{kj} = \sum_{k \in I} \Xi_i \Lambda^*_k \Lambda_k \Xi^*_j = \Xi_i \left( \sum_{k \in I} \Lambda^*_k \Lambda_k \right) \Xi^*_j = \Xi_i S_{\Lambda} \Xi^*_j = S_{ij} = [S_{\Lambda}]_{ij}.$$ 

Thus, $\Lambda^* \Lambda = S_{\Lambda}$, where $\Lambda^* \Lambda = S_{\Lambda}$ means that $\Lambda^* \Lambda = [S_{\Lambda}]$. 

The following result is a generalization of [4, Proposition 3] to g-frames about dependence of the g-R-dual sequence \( \{ \Gamma_j^\Lambda \} \) to choose the g-orthonormal bases \( \Xi = \{ \Xi_j \}_{j \in I} \) and \( \Psi = \{ \Psi_j \}_{j \in I} \).

**Theorem 2.5.** Let \( \Xi = \{ \Xi_j \}_{j \in I} \), \( \Xi' = \{ \Xi'_j \}_{j \in I} \) and \( \Psi = \{ \Psi_j \}_{j \in I} \), \( \Psi' = \{ \Psi'_j \}_{j \in I} \) be g-orthonormal bases for \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in I} \) and \( \{ V_j \}_{j \in I} \) and let \( \Lambda = \{ \Lambda_j \}_{j \in I} \) be a g-Bessel sequence for \( \mathcal{H} \) with respect to \( \{ V_j \}_{j \in I} \). Denote the analysis matrix for \( \Lambda \) with respect to \( \Xi \) by \( A \) and the g-R-dual sequences of \( \Lambda \) with respect to \( (\Xi, \Psi) \) and \( (\Xi', \Psi') \) by \( \{ \Gamma_j^\Lambda \}_{j \in I} \), \( \{ \Gamma_j^{\Lambda'} \}_{j \in I} \), respectively. Then the following conditions are equivalent.

(i) \( \Gamma_j^\Lambda = \Gamma_j^{\Lambda'} \) for all \( j \in I \).

(ii) If \( B \) and \( C \) are the transition matrices from \( \Xi \) to \( \Xi' \) and \( \Psi \) to \( \Psi' \), respectively, then \( AB^* = CA \).

**Proof.** Let \( B = [B_{ij}] \) and \( C = [C_{ij}] \). By the definition of \( \{ \Gamma_j^\Lambda \}_{j \in I} \), \( \{ \Gamma_j^{\Lambda'} \}_{j \in I} \) for every \( i, j \in I \) we have

\[
\Psi_i(\Gamma_j^\Lambda)^* = \lambda_i \xi_i^* \quad \text{and} \quad \Psi_i(\Gamma_j^{\Lambda'})^* = \lambda_i \xi_i'^*.
\]

Since

\[
[AB^*]_{ij} = \sum_{k \in I} \Lambda_{ik} B_{kj}^* = \sum_{k \in I} \lambda_i \xi_i^* \xi_k^* \xi_k'^* = \sum_{k \in I} \left( \sum_{k \in I} \xi_i^* \xi_k^* \right) \xi_k'^* = \lambda_i \xi_i'^* = \Psi_i(\Gamma_j^{\Lambda'})^*.
\]

The conclusion follows. \( \square \)

**Corollary 2.6.** In addition to the hypothesis of Theorem 2.5, if \( \Lambda = \{ \Lambda_j \}_{j \in I} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{ V_j \}_{j \in I} \) and \( \{ \Gamma_j^\Lambda \}_{j \in I} = \{ \Gamma_j^{\Lambda'} \}_{j \in I} \), then \( A^* C^* A S_{\Lambda}^{-1} B^* = I \), where \( I \) is the identity matrix.

**Proof.** Let \( \Lambda = \{ \Lambda_j \}_{j \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{ V_j \}_{j \in I} \). Definition 2.4 implies that \( S_{\Lambda}^{-1} A^* \Lambda A = I \). Thus, if \( \Gamma_j^\Lambda = \Gamma_j^{\Lambda'} \) for all \( j \in I \), then by Theorem 2.5, \( AB^* = CA \). This implies \( B^* = S_{\Lambda}^{-1} A^* CA \). But \( B \) has to be unitary, which yields \( A^* C^* A S_{\Lambda}^{-1} B^* = I \). \( \square \)

Recall that two sequences \( \{ \Gamma_j \}_{j \in I} \) and \( \{ \Gamma_j^{\Lambda'} \}_{j \in I} \) are called equivalent (unitarily equivalent) in \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in I} \), if there exists a bounded linear invertible (unitary) operator \( \Gamma : \mathcal{H} \to \mathcal{H} \) such that \( \Gamma \Gamma_j^\Lambda = \Gamma_j^{\Lambda'} \) for all \( j \in I \).

To have a better understanding of the different types of equivalency, we prove the following characterization result.

**Theorem 2.7.** In addition to the hypothesis of Theorem 2.5, if \( \Gamma = \{ \Gamma_j^\Lambda \}_{j \in I} \) and \( \Gamma' = \{ \Gamma_j^{\Lambda'} \}_{j \in I} \) are g-frames for \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in I} \) and \( \{ V_j \}_{j \in I} \), respectively, then the following statements hold.

(i) If \( \Lambda = \{ \Lambda_j \}_{j \in I} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{ V_j \}_{j \in I} \), then \( \{ \Gamma_j^\Lambda \}_{j \in I} \) is equivalent to \( \{ \Gamma_j^{\Lambda'} \}_{j \in I} \) in \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in I} \) if and only if \( \ker(\Lambda) = \ker(AB^*) \).

(ii) \( \{ \Gamma_j^\Lambda \}_{j \in I} \) is unitarily equivalent to \( \{ \Gamma_j^{\Lambda'} \}_{j \in I} \) in \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in I} \), if and only if

\[
A^* \Lambda = (AB^*)^*(AB^*).
\]

Moreover, if \( \Lambda = \{ \Lambda_j \}_{j \in I} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{ V_j \}_{j \in I} \), then the above is equivalent to \( S_{\Lambda} = B S_{\Lambda} B^* \).

**Proof.**

(i) First we observe that, for every \( g' = \{ g'_k \}_{k \in I} \in \left( \sum_j \oplus V_j \right)_\ell \), we have

\[
\sum_{k \in I} \|g'_k\|^2 = \sum_{k \in I} \langle g'_k, g'_k \rangle = \sum_{k \in I} \sum_{i \in I} \Psi'_k \Psi'_i g'_k g'_k = \sum_{i \in I} \sum_{k \in I} \Psi'_i g'_k g'_k = \sum_{k \in I} \| \Psi'_k g'_k \|^2.
\]
Therefore,
\[ \sum_{k \in I} \psi'_k g'_k = 0 \iff g' = 0. \]

(Necessity). Suppose that \( \{\Gamma'_j^A\}_{j \in I} \) is equivalent to \( \{\Gamma^A_j\}_{j \in I} \) in \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \), then there exists a bounded linear invertible operator \( T : \mathcal{H} \to \mathcal{H} \) such that
\[
T\left( \sum_{j \in I} (\Gamma'_j^A)^* g_j \right) = \sum_{j \in I} (\Gamma_j^A)^* g_j, \quad \forall \{g_j\}_{j \in I} \in \left( \sum_{j \in I} \oplus W_j \right)_{\mathcal{F}}.
\]
Now, \( Ag = 0 \) with \( g = \{g_j\}_{j \in I} \), if and only if
\[
T^{-1}\left( \sum_{j \in I} (\Gamma'_j^A)^* g_j \right) = \sum_{j \in I} (\Gamma'_j^A)^* g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k \Xi'_{j}^* g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k g_j = \sum_{k \in I} \psi'_k (Ag)_k = 0,
\]
if and only if
\[
\sum_{k \in I} \psi'_k (AB^* g) = \sum_{k \in I} \psi'_k \left( \sum_{j \in I} [AB^*]_{kj} g_j \right) = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k B^*_{kj} g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k \Xi'_{j}^* g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k \Xi'_{j}^* g_j = \sum_{j \in I} (\Gamma'_j^A)^* g_j = \sum_{j \in I} (\Gamma_j^A)^* g_j = 0,
\]
if and only if \( AB^* g = 0 \).

(Sufficiency). Suppose that \( \ker(A) = \ker(AB^*) \). Define the operator \( T \) as follows:
\[
T : \text{span} \{ (\Gamma_j^A)^* (W_j) \}_{j \in I} \to \text{span} \{ (\Gamma'_j^A)^* (W_j) \}_{j \in I}, \quad T\left( \sum_{j \in I} (\Gamma_j^A)^* g_j \right) = \sum_{j \in I} (\Gamma'_j^A)^* g_j,
\]
for all \( J \subset I \) with \( |J| < \infty \) and \( g_j \in W_j (j \in J) \). Let \( C, D > 0 \) be the g-frame bounds for g-frame \( \Lambda = \{\Lambda_i\}_{i \in I} \). Then we have
\[
\| T\left( \sum_{j \in J} (\Gamma'_j^A)^* g_j \right) \|^2 = \sum_{j \in J} \| (\Gamma'_j^A)^* g_j \|^2 = \sum_{k \in I} \sum_{j \in J} \| \psi'_k \Lambda_k \Xi'_{j}^* g_j \|^2 \leq D \sum_{j \in J} \| \Xi'_{j}^* g_j \|^2 = D \sum_{j \in J} \| g_j \|^2 = D \sum_{k \in I} \sum_{j \in J} \| \psi'_k \Lambda_k \Xi'_{j}^* g_j \|^2 \leq \frac{D}{C} \sum_{k \in I} \| \Lambda_k \Xi'_{j}^* g_j \|^2 = \frac{D}{C} \sum_{k \in I} \sum_{j \in J} \| \psi'_k \Lambda_k \Xi'_{j}^* g_j \|^2 \leq \frac{D}{C} \sum_{j \in J} \| (\sum_{k \in I} \Xi'_{j}^* \Lambda_k \psi_k)^* g_j \|^2 = \frac{D}{C} \sum_{j \in J} \| (\Gamma'_j^A)^* g_j \|^2.
\]
This shows that \( T \) is a bounded linear operator. To prove invertibility of \( T \) we compute
\[
T\left( \sum_{j \in I} (\Gamma'_j^A)^* g_j \right) = \sum_{j \in I} (\Gamma'_j^A)^* g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k \Xi'_{j}^* g_j = \sum_{k \in I} \sum_{j \in I} \psi'_k \Lambda_k \left( \sum_{i \in I} \Xi_{j}^* \Xi'_{i}^* g_j \right) = \sum_{k \in I} \psi'_k \left( [AB^*]_{kj} g_j \right) = \sum_{k \in I} \psi'_k (AB^* g)_k.
\]
We also have
\[
\sum_{j \in I} (\Gamma_j^A)^* g_j = \sum_{k \in I} \sum_{j \in I} \psi_k^* \Lambda_k \Xi_j^* g_j = \sum_{k \in I} \psi_k^* (A g)_k.
\]

Hence,
\[
T\left( \sum_{j \in I} (\Gamma_j^A)^* g_j \right) = 0 \Leftrightarrow \sum_{j \in I} (\Gamma_j^A)^* g_j = 0.
\]

This implies that \( T \) is invertible operator. Now, the g-completeness of \( \Gamma \) and \( \Gamma' \) for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) implies that \( T \) has an extension invertible on \( \mathcal{H} \) and \( T(\Gamma_j^A)^* = (\Gamma_{j'}^{A'})^* \) for all \( j \in I \).

(ii) First, we prove \( [A^* A]_{ij} = \Gamma_j^A(\Gamma_j^A)^* \) and \( (AB^*)^* (AB^*) \) for all \( j \in I \).

\[
\Gamma_j^A = \left( \sum_{k \in I} \Xi_k^* \psi_k \right) \left( \sum_{m \in I} \psi_m^* \Lambda_m \Xi_j^* \right) = \sum_{k \in I} \sum_{m \in I} \delta_{km} \Xi_k^* \Lambda_m \Xi_j^*\]
\[
= \sum_{k \in I} \sum_{m \in I} \delta_{km} \Xi_k^* \Lambda_m \Xi_j^* = \sum_{k \in I} \Xi_k^* (A^* A)_{ij} = (AB^*)^* (AB^*)_{ij}.
\]

Moreover, we obtain
\[
\Gamma_j^A(\Gamma_j^A)^* = \left( \sum_{k \in I} \Xi_k^* \psi_k \right) \left( \sum_{m \in I} \psi_m^* l_{\Lambda_m} \Xi_j^* \right) = \sum_{k \in I} \sum_{m \in I} \delta_{km} \Xi_k^* \Lambda_m \Xi_j^*\]
\[
= \sum_{k \in I} \sum_{m \in I} \delta_{km} \Xi_k^* \Lambda_m \Xi_j^* = \sum_{k \in I} \Xi_k^* (A^* A)_{ij} = (AB^*)^* (AB^*)_{ij}.
\]

Now, let \( A^* A = (AB^*)^* (AB^*) \). Define the operator \( T \) as follows:
\[
T : \text{span} \{ (\Gamma_j^A)^*(W_j) \}_{j \in I} \to \text{span} \{ (\Gamma_{j'}^{A'})^*(W_j) \}_{j \in I'}, T(\sum_{j \in I} (\Gamma_j^A)^* g_j) = \sum_{j \in I} (\Gamma_j^A)^* g_j,
\]

for all finite subsets \( J \subset I \) and \( g_j \in W_j \) (\( j \in J \)). Let \( f_1, f_2 \in \text{span} \{ (\Gamma_j^A)^*(W_j) \}_{j \in I} \) as \( f_1 = \sum_{j \in J_1} (\Gamma_j^A)^* g_{1j} \) and \( f_2 = \sum_{j \in J_2} (\Gamma_j^A)^* g_{2j} \), we have
\[
\langle Tf_1, Tf_2 \rangle = \sum_{j \in J_1} (\Gamma_j^A)^* g_{1j} \sum_{k \in J_2} (\Gamma_k^A)^* g_{2k} = \sum_{j \in J_1} \sum_{k \in J_2} \langle \Gamma_j^A, \Gamma_k^A \rangle g_{1j} g_{2k} = \langle f_1, f_2 \rangle.
\]

This implies that \( T \) is a bounded linear surjective isometry operator. Thus, the g-completeness of \( \Gamma \) and \( \Gamma' \) for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) implies that \( T \) has an extension isometry on \( \mathcal{H} \) and \( T(\Gamma_j^A)^* = (\Gamma_{j'}^{A'})^* \) for all \( j \in I \). This shows that \( \Gamma \) is unitarily equivalent to \( \Gamma' \) in \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \). The converse implication is obvious. Finally, if \( A = \{A_i\}_{i \in I} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \), then, since \( A^* A = S_A \), thus
\[
S_A = A^* A = (AB^*)^* (AB^*) = BA^* AB^* = B S_A B^*.
\]
3. Characterizations of equivalence of the g-R-dual sequence

In this section we first characterize all sequences with lower g-frame bound. Next, we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. We also characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

Recall that a family \( \{ \Lambda_i \}_{i \in I} \) is a g-frame sequence with respect to \( \{ V_i \}_{i \in I} \), if it is a g-frame for \( \operatorname{span}(\Lambda_i^* \{ V_i \})_{i \in I} \) with respect to \( \{ V_i \}_{i \in I} \).

There exists a characterization of frames which keeps the information about the frame bounds ([5, Lemma 5.5.5]). A similar result holds in g-frame situation.

**Proposition 3.1.** Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, V_i) : i \in I \} \). Then the following conditions are equivalent.

(i) \( \Lambda = \{ \Lambda_i \}_{i \in I} \) is a g-frame sequence with respect to \( \{ V_i \}_{i \in I} \) with g-frame bounds \( A \) and \( B \).

(ii) The synthesis operator \( T^*_\Lambda \) is well-defined on \( (\sum_{i \in I} \oplus V_i)_{\ell^2} \) such that:

\[
\Lambda \| g' \|^2_2 \leq \| T^*_\Lambda g' \|^2_2 \leq B \| g' \|^2_2, \quad \forall \ g' \in (\ker T^*_\Lambda)^\perp.
\]

**Proof.** This follows immediately from [5, Lemma 5.5.5]. \( \square \)

The next result shows a basic connection between a sequence of operators and its g-R-dual sequence which will be used frequently in what follows.

**Theorem 3.2.** Let \( \Lambda = \{ \Lambda_i \}_{i \in I} \) be a g-Bessel sequence for \( \mathcal{H} \) with respect to \( \{ V_i \}_{i \in I} \). Then for every \( \{ g_j \}_{j \in J} \in (\sum_{j \in J} \oplus W_j)_{\ell^2} \), \( \{ \Lambda^*_i \}_{i \in I} \) satisfying \( f = \sum_{j \in J} \Xi_j \Lambda^*_i g_j \) and \( h = \sum_{j \in J} \Psi^*_j g_j \), we have

\[
\left\| \sum_{j \in J} (\Gamma_j^\Lambda)^* g_j \right\|^2 = \sum_{i \in I} \| \Lambda^*_i f \|^2 \quad \text{and} \quad \left\| \sum_{i \in I} \Lambda^*_i g_i \right\|^2 = \sum_{j \in J} \| \Gamma_j^\Lambda h \|^2.
\]

**Proof.** It is easy to check that

\[
\left\| \sum_{j \in J} (\Gamma_j^\Lambda)^* g_j \right\|^2 = \left\| \sum_{j \in J} \left( \sum_{i \in I} \Xi_j \Lambda_i^* \Psi_i^* \right)^* g_j \right\|^2 = \left\| \sum_{i \in I} \Psi_i^* \Lambda_i f \right\|^2 = \sum_{i \in I} \sum_{j \in J} \langle \Lambda_i f, \Psi_i^* \Psi_j^* \Lambda_j f \rangle
\]

\[
= \sum_{i \in I} \sum_{j \in J} \langle \Lambda_i f, \delta_{ij} \Lambda_j f \rangle = \sum_{i \in I} \| \Lambda_i f \|^2.
\]

Similarly, the second claim follows from Theorem 2.3. \( \square \)

**Corollary 3.3.** Let \( \Lambda = \{ \Lambda_i \}_{i \in I} \) be a g-Bessel sequence for \( \mathcal{H} \) with respect to \( \{ V_i \}_{i \in I} \). Then

\[
\| T^*_{\Gamma \Lambda}([f]_{\Xi}) \| = \| T_{\Lambda} f \|_{\ell^2}, \quad \| T^*_{\Lambda}([f]_{\Psi}) \| = \| T_{\Gamma \Lambda} f \|_{\ell^2},
\]

for every \( f \in \mathcal{H} \).

**Proof.** This follows immediately from Theorem 3.2. \( \square \)

There exists an interesting relation between the synthesis operator of \( \Lambda = \{ \Lambda_i \}_{i \in I} \) and the span of \( \{(\Gamma_j^\Lambda)^* \{ W_j \}) \}_{j \in J} \), which will turn out to be very useful in the sequel.

**Theorem 3.4.** Let \( \Lambda = \{ \Lambda_i \}_{i \in I} \) be a g-Bessel sequence for \( \mathcal{H} \) with respect to \( \{ V_i \}_{i \in I} \) with g-R-dual sequence \( \{ \Gamma_j^\Lambda \}_{j \in J} \) with respect to \( (\Xi, \Psi) \). Then the following statements hold.

(i) \( f \in (\operatorname{span}(\{(\Gamma_j^\Lambda)^* \{ W_j \}) \}_{j \in J})^\perp \) if and only if \( [f]_{\Psi} \in \ker T^*_\Lambda \).
(ii) \( f \in (\text{span}(\Lambda^*_i(V_j)))_{i \in \mathbb{I}} \perp \) if and only if \([f]_\Xi \in \ker T^*_\Lambda\).

**Proof.** Let \( f \in \mathcal{H} \). First for each \( j \in \mathbb{J} \) and \( g_j \in W_j \) we observe that

\[
(f, (\Gamma_j^\Lambda)^* g_j) = \sum_{i \in \mathbb{I}} (f, \Psi_i^* \Lambda_i \Xi_j^* g_j) = \sum_{i \in \mathbb{I}} \Lambda_i^* \Psi_i^* f, \Xi_j^* g_j = \langle T^*_\Lambda([f]_\Psi), \Xi_j^* g_j \rangle.
\]

Since \( \Xi = \{\Xi_j\}_{j \in \mathbb{J}} \) is a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in \mathbb{J}} \), \( \langle T^*_\Lambda([f]_\Psi), \Xi_j^* g_j \rangle = 0 \) for all \( j \in \mathbb{I} \) and \( g_j \in W_j \), if and only if \( T^*_\Lambda([f]_\Psi) = 0 \). Thus, \( f \in (\text{span}((\Gamma_j^\Lambda)^*(W_j)))_{j \in \mathbb{J}} \perp \) is equivalent to \([f]_\Psi \in \ker T^*_\Lambda\).

Similarly, the second claim follows from Theorem 2.3.

**Corollary 3.5.** Let \( \Lambda = \{\Lambda_i\}_{i \in \mathbb{I}} \) be a g-Bessel sequence for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in \mathbb{I}} \) with g-R-dual sequence \( \{\Gamma_j^\Lambda\}_{j \in \mathbb{J}} \) with respect to \( (\Xi, \Psi) \). Then

\[
\text{dim} (\text{span}((\Gamma_j^\Lambda)^*(W_j)))_{j \in \mathbb{J}} \perp = \text{dim} \ker T^*_\Lambda \quad \text{and} \quad \text{dim} (\text{span}(\Lambda_i^*(V_j)))_{j \in \mathbb{J}} \perp = \text{dim} \ker T^*_\Lambda.
\]

**Proof.** This follows immediately from Theorem 3.4.

The next result shows a kind of equilibrium between a sequence of operators and its R-dual sequence. It can be viewed as a general version of [4, Proposition 13].

**Corollary 3.6.** The following conditions are equivalent.

(i) \( \Lambda = \{\Lambda_i\}_{i \in \mathbb{I}} \) is a g-frame sequence with respect to \( \{V_i\}_{i \in \mathbb{I}} \) with g-frame bounds \( A, B \).

(ii) \( \{\Gamma_j^\Lambda\}_{j \in \mathbb{J}} \) is a g-frame sequence with respect to \( \{W_j\}_{j \in \mathbb{J}} \) with g-frame bounds \( A, B \).

(iii) \( \{\Gamma_j^\Lambda\}_{j \in \mathbb{J}} \) is a g-Riesz basic sequence with respect to \( \{W_j\}_{j \in \mathbb{J}} \) with g-frame bounds \( A, B \).

**Proof.** (i) \( \iff \) (ii). The Proposition 3.1 and Theorem 3.4 conclude that \( \Lambda = \{\Lambda_i\}_{i \in \mathbb{I}} \) is a g-frame sequence with respect to \( \{V_i\}_{i \in \mathbb{I}} \) with g-frame bounds \( A, B \) if and only if

\[
A \|f\|_2^2 \leq \|T^*_\Lambda([f]_\Psi)\|_2^2 \leq B \|f\|_2^2,
\]

for all \( f \in \text{span}((\Gamma_j^\Lambda)^*(W_j))_{j \in \mathbb{J}} \). Now, Corollary 3.3 implies

\[
A \|f\|_2^2 \leq \|T^*_\Lambda f\|_2^2 \leq B \|f\|_2^2.
\]

(i) \( \iff \) (iii). This equivalence follows immediately from Theorem 3.2.

The dimension condition in Corollary 3.5 will play a crucial role for the g-R-dual sequence. Using Corollary 3.5 we can derive a simple characterization of a g-Riesz basic sequence being a g-R-dual sequence of a g-frame in the tight case.

**Theorem 3.7.** Let \( \Lambda = \{\Lambda_i\}_{i \in \mathbb{I}} \) be a A-tight g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in \mathbb{I}} \) and let \( \{\Gamma_j\}_{j \in \mathbb{J}} \) be an A-tight g-Riesz basic sequence in \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in \mathbb{J}} \). Then \( \{\Gamma_j\}_{j \in \mathbb{J}} \) is a g-R-dual sequence of \( \{\Lambda_i\}_{i \in \mathbb{I}} \) with respect to \( (\Xi, \Psi) \), if and only if

\[
\text{dim} (\text{span}(\Gamma_j^*(W_j)))_{j \in \mathbb{J}} \perp = \text{dim} \ker T^*_\Lambda.
\]

**Proof.** The necessity of the condition in (3.1) follows from Corollary 3.5. Now, assume that (3.1) holds. Then, according to Lemma 1.6 the sequence \( \{\sqrt{\Lambda^*_i} \}\}_{i \in \mathbb{I}} \) is a g-orthonormal system for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in \mathbb{J}} \). Suppose that \( \Xi = \{\Xi_j\}_{j \in \mathbb{J}} \) and \( \Psi = \{\Psi_j\}_{j \in \mathbb{J}} \) are g-orthonormal bases for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in \mathbb{J}} \) and \( \{V_i\}_{i \in \mathbb{I}} \), respectively. Consider the g-R-dual \( \{\Theta_j\}_{j \in \mathbb{J}} \) of \( \Lambda = \{\Lambda_i\}_{i \in \mathbb{I}} \) with respect to \( (\Xi, \Psi) \), i.e., \( \Theta_j = \sum_{i \in \mathbb{I}} \Xi_i^* \Psi_j^* \Lambda_i^* \), \( j \in \mathbb{J} \). By Corollary 3.6 \( \{\Theta_j\}_{j \in \mathbb{J}} \) is an A-tight g-Riesz basic sequence with respect to
\{W_j\}_{j \in I} and hence \{\frac{1}{\sqrt{A}} \Theta_j\}_{j \in I} is also a g-orthonormal system for \mathcal{K} with respect to \{W_j\}_{j \in I}. By Corollary 3.5 and (3.1),
\[
\dim \left( \text{span}(\Theta_j^*(W_j))_{j \in I} \right)^\perp = \dim \ker T^*_A = \dim \left( \text{span}(\Gamma_j^*(W_j))_{j \in I} \right)^\perp.
\] (3.2)
In case \(\left( \text{span}(\Theta_j^*(W_j))_{j \in I} \right)^\perp = \left( \text{span}(\Gamma_j^*(W_j))_{j \in I} \right)^\perp = \{0\}\), the g-orthonormality of the sequences \{\frac{1}{\sqrt{A}} \Theta_i\}_{i \in I} and \{\frac{1}{\sqrt{A}} \Gamma_i\}_{i \in I} implies that there exists unitary operator
\[U : \mathcal{K} \to \mathcal{K}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \quad \forall j \in I.\]
In case \(\left( \text{span}(\Theta_j^*(W_j))_{j \in I} \right)^\perp \neq \{0\}\), letting \{\Phi_j\}_{j \in I} and \{\Omega_j\}_{j \in I} be g-orthonormal bases for
\[\left( \text{span}(\Theta_j^*(W_j))_{j \in I} \right)^\perp \quad \text{and} \quad \left( \text{span}(\Gamma_j^*(W_j))_{j \in I} \right)^\perp,
\]
with respect to \{W_j\}_{j \in I}, respectively, (3.2) implies that there exists unitary operator
\[U : \mathcal{K} \to \mathcal{K}, \quad \text{by} \quad \Gamma_j = \Theta_j U^*, \quad \Omega_j = \Phi_j U^* \quad \forall j \in I.\]
In both cases, we have
\[\Gamma_j = \Theta_j U^* = \left( \sum_{i \in I} \Xi_j^* \Lambda_i^* \Psi_i \right) U^* = \sum_{i \in I} \Xi_j^* \Lambda_i^* \Psi_i U^*, \quad \forall j \in I,
\]
which shows that \{\Gamma_j\}_{j \in I} is a g-R-dual sequence of \{\Lambda_i\}_{i \in I} with respect to \{\Xi_j\}_{j \in I} and \{\Psi_i U^*\}_{i \in I}.

The following result is about different types of equivalence of g-frames, which is taken from [12]. This result will moreover be employed in several proofs in the sequel.

**Proposition 3.8.** Let \(\Lambda = \{\Lambda_i\}_{i \in I}\) and \(\Lambda' = \{\Lambda'_i\}_{i \in I}\) be Parseval g-frames for \(\mathcal{H}_1\) and \(\mathcal{H}_2\) with respect to \(\{V_i\}_{i \in I}\), respectively. Then \(\Lambda\) is unitarily equivalent to \(\Lambda'\) if and only if the analysis operators \(T\Lambda\) and \(T\Lambda'\) have the same range. Likewise, two g-frames with respect to \(\{V_i\}_{i \in I}\) are equivalent if and only if their analysis operators have the same range.

In the following we characterize those pairs of g-frames and their g-R-dual sequences, which are equivalent (unitarily equivalent).

**Theorem 3.9.** Let \(\Lambda = \{\Lambda_i\}_{i \in I}\) and \(\Lambda' = \{\Lambda'_i\}_{i \in I}\) be g-frames for \(\mathcal{H}\) with respect to \(\{V_i\}_{i \in I}\). Then
(i) \(\{\Lambda_i\}_{i \in I}\) is equivalent to \(\{\Lambda'_i\}_{i \in I}\) in \(\mathcal{H}\) with respect to \(\{V_i\}_{i \in I}\) if and only if
\[\text{span}(\Lambda_i^*(W_j))_{j \in I} = \text{span}(\Lambda'_i^*(W_j))_{j \in I};\]
(ii) \(\{\Lambda_i\}_{i \in I}\) is unitarily equivalent to \(\{\Lambda'_i\}_{i \in I}\) in \(\mathcal{H}\) with respect to \(\{V_i\}_{i \in I}\) if and only if \(S_{\Gamma\Lambda} = S_{\Gamma\Lambda'}\);
(iii) \(\{\Gamma_i\}_{i \in I}\) is unitarily equivalent to \(\{\Gamma'_i\}_{i \in I}\) in \(\mathcal{H}\) with respect to \(\{W_i\}_{i \in I}\) if and only if \(S_{\Gamma\Lambda} = S_{\Gamma\Lambda'}.

**Proof.**
(i) By Proposition 3.8, \(\{\Lambda_i\}_{i \in I}\) and \(\{\Lambda'_i\}_{i \in I}\) are equivalent in \(\mathcal{H}\) with respect to \(\{V_i\}_{i \in I}\), if and only if \(R_{T\Lambda} = R_{T\Lambda'}\), and hence \(ker T^*_A = ker T^*_{A'}\). Now the claim follows from Theorem 3.4.
(ii) Using Propositions 3.1 and 3.8, \(\{\Lambda_i\}_{i \in I}\) is unitarily equivalent to \(\{\Lambda'_i\}_{i \in I}\) if and only if
\[
\| \sum_{i \in I} \Lambda_i g_i \|^2 = \| \sum_{i \in I} \Lambda'_i g'_i \|^2, \quad \forall (g'_i)_{i \in I} \in (ker T^*_A)^\perp.
\]
By Theorem 3.2, this in turn is equivalent to
\[
\langle S_{\Gamma\Lambda} f, f \rangle = \sum_{j \in I} ||\Gamma_j^* f||^2 = \sum_{j \in I} ||\Gamma'_j^* f||^2 = \langle S_{\Gamma\Lambda'} f, f \rangle,
\]
for all \(f \in \mathcal{H}\) and \(g'_i = \Psi_i f \quad (i \in I)\). It follows that \(S_{\Gamma\Lambda} = S_{\Gamma\Lambda'}\), as required.
Proof. This follows immediately from (ii) and Theorem 2.3. □

**Corollary 3.10.** Let \( \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \). Then
\[
\text{span}(\{\Gamma_j^\Lambda\}^*(W_j))_{j \in I} = \text{span}(\{\Gamma_j^\Lambda\}^*(W_j))_{j \in I},
\]
where \( \{\Lambda^*_i\}_{i \in I} \) is the canonical dual g-frame of \( \{\Lambda_i\}_{i \in I} \).

Proof. Since \( \{\Lambda^*_i\}_{i \in I} \) is equivalent to \( \{\Lambda_i\}_{i \in I} \), this claim follows from Theorem 3.9. □

4. Duality properties of the g-R-dual sequence

In this section we characterize all properties of a g-Bessel sequence in terms of properties of their g-R-dual sequence. We will study properties of dual g-frames and canonical dual g-frames. This is a general version of duality principle for g-frames which follows from the Casazza duality relations [4].

The next result gives an explicit form for g-R-dual sequence of the canonical dual g-frame.

**Theorem 4.1.** Let \( \{\Lambda_i\}_{i \in I} \) and \( \{\Omega_i\}_{i \in I} \) be g-frames for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \). Then \( \{\Omega_i\}_{i \in I} \) is a dual g-frame of \( \{\Lambda_i\}_{i \in I} \) if and only if g-R-dual sequences \( \{\Gamma_j^\Omega\}_{j \in I} \) and \( \{\Gamma_j^\Lambda\}_{j \in I} \) are g-biorthogonal, i.e.,
\[
\Gamma_j^\Omega(\Gamma_j^\Lambda)^*g_j = \delta_{ij}g_j, \quad \forall \, i, j \in I, \ g_j \in W_j.
\]

Proof. Let \( \{\Omega_i\}_{i \in I} \) be a dual g-frame of \( \{\Lambda_i\}_{i \in I} \). By definition of \( \{\Gamma_j^\Omega\}_{j \in I} \) and \( \{\Gamma_j^\Lambda\}_{j \in I} \) for every \( i, j \in I \) and \( g_j \in W_j \) we have
\[
\Gamma_j^\Lambda(\Gamma_j^\Omega)^*g_j = \sum_{k \in I} \Xi_i \Lambda^*_k \Psi_k(\sum_{m \in I} \Xi_j \Omega^*_m \Psi_m)^*g_j
= \sum_{k \in I} \Xi_i \Lambda^*_k \Psi_k \Xi^*_j \Omega_k \Psi^*_m \Xi^*_m g_j
= \sum_{k \in I} \Xi_i \Lambda^*_k \Omega_k \Xi^*_j g_j = \Xi_i \left(\sum_{k \in I} \Lambda^*_k \Omega_k \Xi^*_j g_j\right) = \Xi_i \xi^*_j g_j = \delta_{ij}g_j.
\]

The converse implication similarly follows from Theorem 2.3. □

**Corollary 4.2.** Let \( \Lambda = \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \) with canonical dual g-frame denoted by \( \{\Lambda^*_i\}_{i \in I} \). Then the g-R-dual sequences \( \{\Gamma_j^\Lambda\}_{j \in I} \) and \( \{\Gamma_j^\Lambda\}_{j \in I} \) are g-biorthogonal, i.e.,
\[
\Gamma_j^\Lambda(\Gamma_j^\Lambda)^*g_j = \Gamma_j^\Lambda(\Gamma_j^\Lambda)^*g_j = \delta_{ij}g_j
\]
for all \( i, j \in I \) and \( g_j \in W_j \). Thus \( \{\Gamma_j^\Lambda\}_{j \in I} \) is the dual g-Riesz basic sequence of \( \{\Gamma_j^\Lambda\}_{j \in I} \).

The next result is a characterization of tight g-frames in terms of their g-R-dual sequence.

**Corollary 4.3.** \( \{\Lambda_i\}_{i \in I} \) is an A-tight g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \) if and only if g-R-dual sequence \( \{\Gamma_j^\Lambda\}_{j \in I} \) is a g-orthonormal system for \( \mathcal{H} \) with respect to \( \{W_j\}_{j \in I} \). Thus the sequence \( \{\Lambda_i\}_{i \in I} \) is a Parseval g-frame if and only if, its g-R-dual sequence is an orthonormal system.

Proof. This follows immediately from Lemma 1.6, Corollary 3.6, and Theorem 4.2. □

**Theorem 4.4.** Let \( \{\Lambda_i\}_{i \in I} \) and \( \{\Omega_i\}_{i \in I} \) be g-frames for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \). Then \( \{\Omega_i\}_{i \in I} \) is a dual g-frame of \( \{\Lambda_i\}_{i \in I} \) if and only if, there exists a g-Bessel sequence \( \{\Theta_j\}_{j \in I} \) for \( \text{span}(\{\Gamma_j^\Lambda\}^*(W_j))_{j \in I} \perp \) with respect to \( \{W_j\}_{j \in I} \), such that \( \Gamma_j^\Omega = \Gamma_j^\Lambda + \Theta_j \) for all \( j \in I \).
Proof. Suppose that \( \{\Omega_i\}_{i \in I} \) is a dual g-frame of \( \{\Lambda_i\}_{i \in I} \). By Theorem 4.1 we have
\[
\langle (\Gamma_i^\Omega - \Gamma_i^\Lambda)^* g_i, (\Gamma_j^\Lambda)^* g_j \rangle = \langle g_i, (\Gamma_i^\Omega - \Gamma_i^\Lambda)(\Gamma_j^\Lambda)^* g_j \rangle - \langle g_i, \Gamma_i^\Lambda(\Gamma_j^\Lambda)^* g_j \rangle
= \langle g_i, \delta_{ij} g_j \rangle - \langle g_i, \delta_{ij} g_j \rangle = 0,
\]
for all \( i, j \in I \) and \( g_i \in W_i, g_j \in W_j \). Thus, Definition 2.1 implies that \( \Theta_i = \Gamma_i^\Omega - \Gamma_i^\Lambda \) is a g-Bessel sequence for \( (\text{span}(\langle \Gamma_j^\Lambda \rangle^* W_j))_{j \in I}^\perp \) with respect to \( \{W_j\}_{j \in I} \) and \( \Gamma_i^\Omega = \Gamma_i^\Lambda + \Theta_i \). Now for the opposite implication, suppose that there exists a g-Bessel sequence \( \{\Theta_i\}_{i \in I} \) for \( (\text{span}(\langle \Gamma_j^\Lambda \rangle^* W_j))_{j \in I}^\perp \) with respect to \( \{W_j\}_{j \in I} \), such that \( \Gamma_i^\Omega = \Gamma_i^\Lambda + \Theta_i \) for all \( i \in I \). By Theorem 2.3, we have
\[
\Omega_i = \hat{\Lambda}_i + \sum_{j \in I} \Psi_i(\Theta_j)^* \zeta_j \quad \text{for all } i \in I.
\]
So, for each \( f \in \mathcal{H} \)
\[
\sum_{i \in I} \Lambda_i^* \Omega_i f = \sum_{i \in I} \Lambda_i^* (\hat{\Lambda}_i + \sum_{j \in I} \Psi_i(\Theta_j)^* \zeta_j) f = \sum_{i \in I} \Lambda_i^* \hat{\Lambda}_i f + \sum_{i \in I} \sum_{j \in I} \Lambda_i^* \Psi_i(\Theta_j)^* \zeta_j f = f + \sum_{j \in I} \sum_{i \in I} \Lambda_i^* \Psi_i(\Theta_j)^* \zeta_j f,
\]
since \( \Theta_j^* \zeta_j f \in (\text{span}(\langle \Gamma_j^\Lambda \rangle^* W_j))_{j \in I}^\perp \) for all \( j \in I \). Theorem 3.4 implies that
\[
\sum_{i \in I} \Lambda_i^* \Psi_i(\Theta_j)^* \zeta_j f = 0.
\]
This proves that \( \{\Omega_i\}_{i \in I} \) is a dual g-frame of \( \{\Lambda_i\}_{i \in I} \). \( \square \)

Among the dual g-frames the canonical dual g-frame is distinguished by the following properties.

**Theorem 4.5.** Let \( \Lambda = \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{V_i\}_{i \in I} \) with canonical dual g-frame denoted by \( \{\hat{\Lambda}_i\}_{i \in I} \) and let \( \Omega_i \) be a dual g-frame of \( \{\Lambda_i\}_{i \in I} \). Then
\[
\|\Gamma_j^\Lambda\| \leq \|\Gamma_j^\Omega\| \quad \text{for all } j \in I,
\]
with equality if and only if \( \{\Omega_j\}_{j \in I} = \{\hat{\Lambda}_j\}_{j \in I} \).

**Proof.** By Theorem 4.4, \( \{\Omega_j\}_{j \in I} \) is a dual g-frame of \( \{\Lambda_i\}_{i \in I} \) if and only if \( \Gamma_j^\Omega = \Gamma_j^\Lambda + \Theta_j \), where \( (\Gamma_j^\Lambda)^* g \in \text{span}(\langle \Gamma_j^\Lambda \rangle^* W_j)_{j \in I} \) and \( \Theta_j^* g \in (\text{span}(\langle \Gamma_j^\Lambda \rangle^* W_j))_{j \in I}^\perp \) for all \( j \in I, g \in W_j \). Hence
\[
\|\Gamma_j^\Omega\|^2 = \|(\Gamma_j^\Omega)^*\|^2 = \sup_{\|g\|=1} \|(\Gamma_j^\Lambda)^* g\|^2 + \sup_{\|g\|=1} \|\Theta_j^* g\|^2 = \|(\Gamma_j^\Lambda)^*\|^2 + \|\Theta_j\|^2 \leq \|\Gamma_j^\Lambda\|^2,
\]
with equality if and only if \( \{\Omega_j\}_{j \in I} = \{\hat{\Lambda}_j\}_{j \in I} \). \( \square \)

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