A sufficient condition for coinciding the Green graphs of semigroups

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Abstract

A necessary condition for coinciding the Green graphs $\Gamma_L(S), \Gamma_R(S), \Gamma_J(S), \Gamma_D(S)$ and $\Gamma_H(S)$ of a finite semigroup $S$ has been studied by Gharibkhajeh [A. Gharibkhajeh, H. Dosstie, Bull. Iranian Math. Soc., 40 (2014), 413–421]. Gharibkhajeh et al. proved that the coinciding of Green graphs of a finite semigroup $S$ implies the regularity of $S$. However, the converse is not true because of certain well-known examples of finite regular semigroups. We look for a sufficient condition on non-group semigroups that implies the coinciding of the Green graphs. Indeed, in this paper we prove that for every non-group quasi-commutative finite semigroup, all of the Green graphs are isomorphic. ©2017 all rights reserved.

Keywords: Quasi-commutativity, finitely presented semigroups, Green relations, Green graphs.

2010 MSC: 20M05, 05C99.

1. Introduction

Let $S$ be a finite semigroup. Following the notation of [4], the left Green Graph $\Gamma_L(S)$ is an undirected graph with vertices $L_i, (1 \leq i \leq t)$ where the $L_i$s are the left Green classes of the semigroup $S$ and two vertices $L_i, L_j$ are adjacent in $\Gamma_L(S)$ if and only if $\gcd(|L_i|, |L_j|) > 1$. These graphs are indeed the generalization of the conjugacy graphs of finite groups studied by Adan-Bante [1]. The right Green graph $\Gamma_R(S)$, the intersection Green graph $\Gamma_J(S)$, the join Green graph $\Gamma_D(S)$, and finally the $\mathcal{J}$-classes Green graph $\Gamma_J(S)$ are defined in a similar way. Investigating these graphs is of interest because of their ability in identifying certain types of finite semigroups, the non-group non-regular quasi-commutative semigroups. As usual, an associative algebraic structure $(S, \cdot)$ is called quasi-commutative if, for every elements $a, b \in S$, there exists a positive integer $r$ such that $ab = b^ra$. For useful information on quasi-commutative semigroups and examples, one may see [2, 3, 5–7]. Our main results on this type of semigroup are the following:

\textbf{Proposition A.} For every non-commutative quasi-commutative semigroup $S$, all Green graphs are isomorphic.

\textbf{Proposition B.} If $b$ is a non idempotent element of a nowhere commutative quasi-commutative finite semigroup $S$, then $b$ is regular if and only if $|\langle b \rangle| > 1$.
Proposition C. Let $S$ be a finite non-regular nowhere commutative quasi-commutative semigroup. Then all of the Green graphs of $S$ are isomorphic to $nK_1 \cup K_m$, where $m$ is the number of $\mathcal{L}$-classes and $n$ is the number of non-regular non idempotent elements of $S$. Moreover, these graphs are not complete.

2. The proofs

We start the proofs with a key lemma.

Lemma 2.1. Every non idempotent regular element $b$ of a finite semigroup $S$ satisfies $||b|_J| > 1$.

Proof. For a regular element $b \in S$, we may use a method of proof similar to the proof of Lemma 1.14. of [3]. Indeed, there exists an element $x \in S$ such that $b = bxb$, and then $y = xbx$ is an inverse for $b$. This yields $yby = (xbx)b(xbx) = x(bxb)(xbx) = x(bxb)x = xbx = y$ and $byb = b(xbx)b = (bxb)xb = bxb = b$. Let $y \neq b$. So, $yby = y$ and $byb = b$ implies that $y \in [b]_J$ and therefore $||b|_J| > 1$. If $y = b$ so $b^3 = b$ and we have the following relations:

$$b = b \cdot b^2 \cdot b^2, \ b^2 = b \cdot b \cdot b^2,$$

which shows that $b^2 \in [b]_J$ and so $||b|_J| > 1$. \qed

Proof of Proposition A. We consider the different cases as follows:

Case 1. $x \mathcal{L} y \Rightarrow x \mathcal{R} y$. If $x \mathcal{L} y$ so, $y = xu$ and $x = yv$, for some $u, v \in S$. Since $S$ is quasi-commutative then there exist integers $r_u$, $r_v$ such that $xy = u^{r_u}x$ and $yv = v^{r_v}y$, respectively. Therefore, the identities $y = u_1x$ and $x = v_1y$ show that $x \mathcal{R} y$, where $u_1 = u^{r_u}$ and $v_1 = v^{r_v}$.

Case 2. $x \mathcal{R} y \Rightarrow x \mathcal{L} y$. In a similar way to the first case and considering the definition of the right Green graphs.

Case 3. $x \mathcal{L} y \iff x \mathcal{R} y$. As in Case 1, $x \mathcal{L} y$ yields $x \mathcal{R} y$. So, by the definition of $\mathcal{H}$-relation, we get $x \mathcal{H} y$. The converse is obvious.

Case 4. $x \mathcal{R} y \iff x \mathcal{J} y$. Similar to Cases 2 and 3.

Case 5. $x \mathcal{L} y \Rightarrow x \mathcal{J} y$. If $x \mathcal{L} y$ then there exist $u, v \in S$ such that $y = xu$ and $x = yv$. Due to the quasi-commutativity of $S$, there exist positive integers $r_v, r_u, r_y, r_x$ such that

$$yv = v^{r_v}y, \ xu = u^{r_u}x, \ vy = y^{r_y}v, \ ux = x^{r_x}u.$$

There are three cases to consider:

1. $r_v > 1, r_y > 1$. We get:

$$x = yv = v^{r_v}y = v^{r_v-1}(yv) = v^{r_v-1}(y^{r_y}v) = (v^{r_v-1})y(y^{r_y-1}v),$$

which yields $x = u_1yv_1$, $(u_1 = v^{r_v-1}, v_1 = y^{r_y-1})$.

2. $r_v = 1, r_y \geq 1$. We get:

$$x = yv = vy = v(xu) = v(yv)u = u_2yv_2, \ (u_2 = v, v_2 = vu).$$

3. $r_v > 1, r_y = 1$. In this situation, we have:

$$x = yv = v^{r_v}y = v^{r_v-1}(yv) = v^{r_v-1}(yv),$$

which yields $x = u_3yv_3$, $(u_3 = v^{r_v-1}, v_3 = v)$. The proof of $y = u_1xy$ for some $u_1, v_j \in S$ is similar.

Case 6. $x \mathcal{R} y \Rightarrow x \mathcal{J} y$. Clearly, $x \mathcal{R} y$ yields $x \mathcal{L} y$ so, $x \mathcal{J} y$. 

Case 7. $x \mathcal{J} y \implies x \mathcal{L} y$. $x \mathcal{J} y$ implies that $x = u_1 y v_1$ and $y = u_2 x v_2$ for some $u_1, u_2, v_1$ and $v_2$ in $S$. Because of the quasi-commutativity of $S$, we have $x = (y^{r_y} u_1) v_1 = y u_2$, $(u_2 = y^{r_y} u_1)$ and $y = x^{r_x} u_2 v_2 = x y$, $(v_3 = x^{r_x} u_2 v_2)$ where $r_y$ and $r_x$ are both positive integers. This shows that $x \mathcal{L} y$.

Case 8. $x \mathcal{D} y \implies x \mathcal{J} y$. Since $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{H}$, then $\mathcal{D} \subseteq \mathcal{L}$. So, the proof is obvious.

Case 9. $x \mathcal{J} y \implies x \mathcal{D} y$. Let $x \mathcal{J} y$. Then, there are elements $u_1, u_2, v_1$ and $v_2$ in $S$ such that

$$x = u_1 y v_1, \quad y = u_2 x v_2.$$ 

Setting $z = u_1 y, k = v_1$ yields $x = u_1 y v_1 = z k$. So, $z = u_1 y = u_1 (u_2 x v_2)$. By the quasi-commutativity of $S$, there are integers $r_1, r_2$ such that $u_2 x = x^{r_1} u_2, u_1 x = x^{r_2} u_1$. Therefore,

$$z = u_1 (u_2 x v_2) = u_1 (x^{r_1} u_2) v_2 = (u_1 x) (x^{r_2-1} u_2 v_2) = x u_3,$$

where, $u_3 = (x^{r_2-1} u_1) (x^{r_2-1} u_2 v_2)$. This shows that $x \mathcal{L} z$. Moreover, $y = u_2 x v_2 = u_2 (z v_3)$, where $v_3 = v_1 v_2$ and so there is an integer $r_{v_3} \geq 1$ such that

$$y = u_2 x v_2 = v_4 z, (v_4 = u_2 ^{r_{v_3}}).$$

The latter identity and $z = u_1 y$ confirm that $z \mathcal{R} y$. This completes the proof of $x \mathcal{D} y$.

Proof of Proposition B. Let $x \mathcal{J} b$ where $x \in S$ and $x \neq b$. So, there exist elements $u_i, v_i \in S, (i = 1, 2)$ such that

$$b = u_1 x v_1, \quad x = u_2 b v_2.$$ 

So we have $b = u_1 (u_2 b v_2) v_1 = u_3 b v_3$ where $u_3 = u_1 u_2, v_3 = v_2 v_1$. Because of the quasi-commutativity of $S$, we can find positive integers $r_b$ such that $u_3 b = b^{r_b} u_3$ and therefore $b = b^{r_b} y$ where $y = u_3 v_3$. Considering two different cases for $r_b$, we have:

1. If $r_b > 1$ so $b = b^{r_b} y = b^{r_b-1} (by)$ and by quasi-commutativity of $S$ we have $b = b^{r_b-1} y^{r_b} b$ where $r_y$ is some positive integer.

2. If $r_b = 1$ then $u_3 b = b u_3$ and so the nowhere commutativity of the semigroup gives $u_3 = b$.

Therefore by the quasi-commutativity of $S$ we have:

$$b = u_3 b v_3 = b \cdot (v_3^{r_{v_3}} b) = b \cdot v_4 \cdot b, (v_4 = v_3^{r_{v_3}}),$$

where $r_{v_3}$ is a positive integer. This means that $b$ is a regular element of $S$. For the converse, we consider Lemma 2.1.

Proof of Proposition C. By using Proposition A, we get that

$$\Gamma_{\mathcal{L}} (S) \cong \Gamma_{\mathcal{R}} (S) \equiv \Gamma_{\mathcal{D}} (S) \equiv \Gamma_{\mathcal{H}} (S).$$

So, identifying the Green graph of $S$ one needs only to consider the $\mathcal{L}$-classes of $S$. If there are $n$ non-regular elements $b_1, b_2, \cdots, b_n \in S$ then by a consequence of Proposition B, we get:

$$nK_1 = \bigcup_{i=1}^{n} \Gamma_{\mathcal{L}} ([b_i]).$$

By considering the set of all $\mathcal{L}$-classes of $S$ as $\{ \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_m \}$, where each class contains at least two elements, we construct the sub-graph $K_m$ of $\Gamma_{\mathcal{L}} (S)$. Consequently,

$$\Gamma_{\mathcal{L}} (S) \equiv nK_1 \cup K_m.$$ 

Since $S$ is non-regular, $\Gamma_{\mathcal{L}} (S)$ is not a complete graph.

Conclusion 2.2. Using a similar proof, we may extend Proposition A for quasi-hamiltonian semigroups. By definition, the semigroup $S$ is quasi-hamiltonian if and only if for every elements $a, b \in S$ there are positive integers $r_a, r_b$ such that $ab = b^{r_b} a^{r_a}$. 

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References


