A Best Proximity Point Theorem in Metric Spaces with Generalized Distance

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Abstract
In this paper at first, we define the weak P-property with respect to a τ-distance such as p. Then we state a best proximity point theorem in a complete metric space with generalized distance such that it is an extension of previous research.

Keywords: weak P-property, best proximity point, τ-distance, weakly contractive mapping, altering distance functions.

1. Introduction
The best proximity point is a interesting topic in best proximity theory. Let A, B be two non-empty subsets of a metric space (X, d) and T: A → B. A solution x, for the equation d(x, Tx) = d(A, B) is called a best proximity point of T. If d(x, Tx) = 0 then x is called a fixed point of T [15]. The existence and convergence of best proximity points has generalized by several authors such as Jleli and Samet [3], Prolla [4], Reich [5], Sadiq Basha [7,8], Sehgal and Singh [10,11], Vertivel, Veermani and Bhattacharyya[13] in many directions. On the other hand Suzuki [12] introduced the concept of τ-distance on a metric space. Many fixed point theorems extended for various contractive mappings with respect to a τ-distance. In this paper, by using the concept of τ-distance, we prove a best proximity point theorem. Our results are extension of a best proximity point theorem in metric spaces.

2. Preliminary
Let A, B be two non-empty subsets of a metric space (X, d). The following notations will be used throughout this paper:
\[ d(y, A) := \inf\{d(x, y): x \in A\}, \]
\[ d(A, B) := \inf\{d(x, y): x \in A, y \in B\}, \]
\[ A_0 := \{x \in A: d(x, y) = d(A, B) \text{ for some } y \in B\}, \]
We recall that $x \in A$ is a best proximity point of the mapping $T: A \to B$ if $d(x, Tx) = d(A, B)$. It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

**Definition 2.1.**[9] Let $(A, B)$ be a pair of non-empty subsets of a metric space $X$ with $A \neq \emptyset$. Then the pair $(A, B)$ is said to have the P-property if and only if
\[
d(x_1, y_1) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2)
\]
where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset $A$ of $X$, the pair $(A, A)$ has the P-property.


**Definition 2.2.**[2] A function $\psi: [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions:
(i) $\psi$ is continuous and non-decreasing.
(ii) $\psi(t) = 0$ if and only if $t = 0$.

**Definition 2.3.**[6] Let $(X, d)$ be a metric space. $T: X \to X$ is weakly contractive if
\[
d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \quad \forall x, y \in X
\]
where $\phi$ is a altering distance function.

Suzuki [12] introduced the concept of $\tau$-distance on a metric space.

**Definition 2.4.**[12] Let $X$ be a metric space with metric $d$. A function $p: X \times X \to [0, \infty)$ is called $\tau$-distance on $X$ if there exist a function $\eta: X \times [0, \infty) \to [0, \infty)$ such that the following are satisfied:
1. $p(x, z) \leq p(x, y) + p(y, z) \quad \forall x, y, z \in X$;
2. $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and $\eta$ is concave and continuous in it’s second variable.
3. $\lim_n x_n = x$ and $\lim_n \sup \{\eta(z_n, p(z_n, x_n)): m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$;
4. $\lim_n \sup \{p(x_n, y_n): m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
5. $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

**Remark 2.5.**[12] It can be replaced $(\tau_2)$ by the following $(\tau_2)'$.

$(\tau_2)'$ $\inf \{\eta(x, t): t > 0\} = 0$ for all $x \in X$ and $\eta$ is non-decreasing in it’s second variable.

**Remark 2.6.** If $(X, d)$ is a metric space, then the metric $d$ is a $\tau$-distance on $X$.

In the following examples, we define $\eta: X \times [0, \infty) \to [0, \infty)$ by $\eta(x, t) = t$, for all $x \in X$ and $t \in [0, \infty)$. It is easy to see that $p$ is a $\tau$-distance on metric space $X$.

**Example 2.7.** Let $(X, d)$ be a metric space and $c$ be a positive real number. Then $p: X \times X \to [0, \infty)$ by $p(x, y) = c$ for $x, y \in X$ is a $\tau$-distance on $X$.

**Example 2.8.** Let $(X, \| \|)$ be a normed space. $p: X \times X \to [0, \infty)$ by $p(x, y) = \| x \| + \| y \|$ for $x, y \in X$ is a $\tau$-distance on $X$.

**Example 2.9.** Let $(X, \| \|)$ be a normed space. $p: X \times X \to [0, \infty)$ by $p(x, y) = \| y \|$ for $x, y \in X$ is a $\tau$-distance on $X$. 

Definition 2.10. 
Let $(X, d)$ be a metric space and $p$ be a $\tau$-distance on $X$. A sequence $\{x_n\}$ in $X$ is a $p$-Cauchy if there exists a function $\eta: X \times [0, \infty) \to [0, \infty)$ satisfying $(\tau 2)-(\tau 5)$ and a sequence $\{z_m\}$ in $X$ such that $\lim_n \sup_{m \geq n} \eta(z_m, p(z_m, x_n)) = 0$.

The following lemmas are essential for the next sections.

Lemma 2.11. 
Let $(X, d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ is a $p$-Cauchy sequence, then it is a Cauchy sequence. Moreover if $\{y_n\}$ is a sequence satisfying $\lim_n \sup_{m \geq n} p(x_m, y_n) = 0$, then $\{y_n\}$ is also $p$-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.

Lemma 2.12. 
Let $(X, d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ in $X$ satisfies $\lim_n p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a $p$-Cauchy sequence. Moreover if $\{y_n\}$ in $X$ also satisfies $\lim_n p(z, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$. In particular, for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.

Lemma 2.13. 
Let $(X, d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If $\{x_n\}$ in $X$ satisfies $\lim_n p(x_n, y_n) = 0$, then $\{x_n\}$ is a $p$-Cauchy sequence. Moreover if $\{y_n\}$ in $X$ satisfies $\lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also $p$-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.

The next result is an immediate consequence of the Lemma 2.11 and Lemma 2.13.

Corollary 2.14. 
Let $(X, d)$ be a metric space and $p$ be a $\tau$-distance on $X$. If a sequence $\{x_n\}$ in $X$ satisfies $\lim_n \sup_{m \geq n} p(x_m, x_n) = 0$, then $\{x_n\}$ is a Cauchy sequence.

3. Main results

Inspire of Sankar Raj [9] and Zhang and others [14], we define the weak $P$-property with respect to a $\tau$-distance as follows:

Definition 3.1. 
Let $(A, B)$ be a pair of non-empty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$. Also let $p$ be a $\tau$-distance on $X$. Then the pair $(A, B)$ is said to have the weak $P$-property with respect to $p$ if and only if

$$d(x_1, y_1) = d(A, B) = p(x_1, x_2) \leq p(y_1, y_2)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset $A$ of $X$, the pair $(A, A)$ has the weak $P$-property with respect to $p$.

Remark 3.2. 
If $p = d$ then $(A, B)$ is said to have the weak $P$-property where $A \neq \emptyset$. (See [14]) It is easy to see that if $(A, B)$ has the $P$-property then $(A, B)$ has the weak $P$-property.

Example 3.3. 
Let $X = \mathbb{R}^2$ with the usual metric and $p_1, p_2$ be two $\tau$-distances that defined in Example 2.8 and Example 2.9, respectively. Consider,

$A = \{(a, b) \in \mathbb{R}^2 | a = 0, 2 \leq b \leq 3\}$,

$B = \{(a, b) \in \mathbb{R}^2 | a = 1, b \leq 1\} \cup \{(a, b) \in \mathbb{R}^2 | a = 1, b \geq 4\}$.

Then $(A, B)$ has the weak $P$-property with respect to $p_1$ and has not the weak $P$-property with respect to $p_2$.

By the definition of $A, B$ we obtain,

$$d((0, 2), (1, 1)) = d((0, 3), (1, 4)) = d(A, B) = \sqrt{2}$$

where $(0, 2), (0, 3) \in A_0$ and $(1, 1), (1, 4) \in B_0$. We have,

$p_1((0, 2), (0, 3)) = 5$ and $p_1((1, 1), (1, 4)) = \sqrt{2} + \sqrt{17}$,

$p_1((0, 3), (0, 2)) = 5$ and $p_1((1, 4), (1, 1)) = \sqrt{17} + \sqrt{2}$.

Therefore $(A, B)$ has the weak $P$-property with respect to $p_1$. On the other hand, we have

$p_2((0, 3), (0, 2)) = 2$ and $p_2((1, 4), (1, 1)) = \sqrt{2}$.

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This implies that \((A, B)\) has not the weak \(P\)-property with respect to \(p_2\).

Sankar Raj\(^9\) stated a best proximity point theorem for weakly contractive non-self mappings in metric spaces. The following Theorem is an extension of his results in a metric spaces with generalized distance.

**Theorem 3.4.** Let \(A\) and \(B\) be non-empty closed subsets of the metric space \((X, d)\) such that \(A_0 \neq \emptyset\). Let \(p\) be a \(\tau\)-distance on \(X\) and \(T: A \rightarrow B\) satisfies the following conditions:

\begin{enumerate}[(a)]
    \item \(T(A_0) \subseteq B_0\) and \((A, B)\) has the the weak \(P\)-property with respect to \(p\).
    \item \(T\) is a continuous function on \(A\) such that 
    \[ \psi(p(Tx, Ty)) \leq \psi(p(x, y)) - \phi(p(x, y)), \quad \forall x, y \in A \]
    where \(\psi\) is an altering distance function and \(\phi: [0, \infty) \rightarrow [0, \infty)\) is non-decreasing function also \(\phi(t) = 0\) if and only if \(t = 0\).
\end{enumerate}

Then \(T\) has a best proximity point in \(A\). Moreover, if \(d(x, Tx) = d(x^*, Tx^*) = d(A, B)\) for some \(x, x^* \in A\), then \(p(x, x^*) = 0\).

**Proof.** Choose \(x_0 \in A_0\). Since \(Tx_0 \in T(A_0) \subseteq B_0\), there exists \(x_1 \in A_0\) such that \(d(x_1, Tx_0) = d(A, B)\). Again, \(Tx_1 \in T(A_0) \subseteq B_0\), there exists \(x_2 \in A_0\) such that \(d(x_2, Tx_1) = d(A, B)\). Continuing this process, we can find a sequence \(\{x_n\}\) in \(A_0\) such that

\[ d(x_{n+1}, Tx_n) = d(A, B), \quad \forall n \in \mathbb{N} \cup \{0\}. \]

\((A, B)\) satisfies the weak \(P\)-property with respect to \(p\), therefore from (1) we obtain,

\[ p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \]

We will prove that the sequence \(\{x_n\}\) is convergent in \(A_0\). Since \(\psi\) is non-decreasing function we receive that

\[ \psi(p(x_n, x_{n+1})) \leq \psi(p(Tx_{n-1}, Tx_n)), \quad \forall n \in \mathbb{N}. \]

Also by the definition of \(T\), we have

\[ \psi(p(Tx_{n-1}, Tx_n)) \leq \psi(p(x_{n-1}, x_n)) - \phi(p(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}. \]

From (3) and (4), we receive that

\[ \psi(p(x_n, x_{n+1})) \leq \psi(p(Tx_{n-1}, Tx_n)) - \phi(p(x_{n-1}, x_n)) \]

\[ \leq \psi(p(x_{n-1}, x_n)) - \phi(p(x_{n-1}, x_n)) \]

\[ \leq \psi(p(x_{n-1}, x_n)), \]

for all \(n \in \mathbb{N}\). Since \(\psi\) is non-decreasing function, we have

\[ p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \]

Therefore, the sequence \(\{p(x_n, x_{n+1})\}\) is monotone non-increasing and bounded. Hence there exists \(r \geq 0\) such that

\[ \lim_{n \to \infty} p(x_n, x_{n+1}) = r \geq 0. \]

We claim that \(r = 0\). Suppose to the contrary, that \(r > 0\). From the inequality

\[ \psi(p(x_n, x_{n+1})) \leq \psi(p(x_{n-1}, x_n)) - \phi(p(x_{n-1}, x_n)) \leq \psi(p(x_{n-1}, x_n)), \]

we obtain

\[ \lim_{n \to \infty} \phi(p(x_{n-1}, x_n)) = 0. \]

Since \(0 < r \leq p(x_n, x_{n+1})\) and \(\phi\) is non-decreasing function,

\[ 0 < \phi(r) \leq \phi(p(x_n, x_{n+1})), \]

So,

\[ 0 < \phi(r) \leq \lim_{n \to \infty} \phi(p(x_n, x_{n+1})), \]

which is a contradiction. Hence \(\lim_{n \to \infty} p(x_n, x_{n+1}) = 0\). Similarly we receive that \(\lim_{n \to \infty} p(x_{n+1}, x_n) = 0\).

Now we show that \(\lim_{n \to \infty} p(x_n, x_m) = 0\) for \(m > n\). In contrary case, there exists \(\epsilon > 0\) and two subsequence \(\{x_{m_k}\}, \{x_{n_k}\}\) such that \(m_k\) is smallest index for which \(m_k > n_k > k, p(x_{n_k}, x_{m_k}) \geq \epsilon\). This means that
Letting $k \to \infty$, we receive that
\begin{equation}
\lim_{k \to \infty} p(x_{nk}, x_{mk}) = \epsilon.
\end{equation}
By triangle inequality, we have
\[
\begin{aligned}
p(x_{nk}, x_{mk}) &\leq p(x_{nk}, x_{nk-1}) + p(x_{nk-1}, x_{mk}) \\
p(x_{nk-1}, x_{mk-1}) &\leq p(x_{nk-1}, x_{nk}) + p(x_{nk}, x_{mk}) + p(x_{mk}, x_{mk-1}).
\end{aligned}
\]
Letting $k \to \infty$ in above two inequality and using (6), we get
\[
\lim_{k \to \infty} p(x_{nk-1}, x_{mk-1}) = \epsilon.
\]
So,
\[
0 < \psi(\epsilon) \leq \psi\left(p(x_{nk}, x_{nk})\right) \\
\leq \psi\left(p(Tx_{nk-1}, Tx_{mk})\right) \\
\leq \psi\left(p(x_{nk-1}, x_{mk})\right) - \phi\left(p(x_{nk-1}, x_{mk})\right) \\
\leq \psi\left(p(x_{nk-1}, x_{mk})\right).
\]
From continuity of $\psi$ in the above inequality, we obtain that
\[
\lim_{k \to \infty} \phi\left(p(x_{nk-1}, x_{mk-1})\right) = 0.
\]
From $\lim_{k \to \infty} p(x_{nk-1}, x_{mk-1}) = \epsilon$, we can find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$,
\[
\epsilon \leq p(x_{nk-1}, x_{mk-1}).
\]
This implies that,
\[
0 < \phi\left(\frac{\epsilon}{2}\right) \leq \phi\left(p(x_{nk-1}, x_{mk-1})\right), \quad \forall k \geq k_0
\]
and this contradicts to (7). Thus $\lim_{n \to \infty} p(x_n, x_m) = 0$ for $m > n$ and this implies that,
\[
\lim_{n \to \infty} \sup\{p(x_n, x_m): m \geq n\} = 0.
\]
Therefore by Corollary 2.14, $\{x_n\}$ is a Cauchy sequence in $A$. Since $X$ is a complete metric space and $A$ is a closed subset of $X$, there exists $x \in A$ such that $\lim_{n \to \infty} x_n = x$. $T$ is continuous, therefore with letting $n \to \infty$ in (1), we obtain
\[
d(x, Tx) = d(A, B).
\]
Now let $x^* \in A$ such that
\[
d(x^*, Tx^*) = d(A, B).
\]
We claim that $p(x, x^*) = 0$. Suppose to the contrary, that $p(x, x^*) > 0$. Hence $\phi(p(x, x^*)) > 0$ and therefore by the definition of $T$, $\psi$, we obtain that,
\[
\psi\left(p(x, x^*)\right) \leq \psi\left(p(Tx, Tx^*)\right) \leq \psi\left(p(x, x^*)\right) - \phi\left(p(x, x^*)\right) \leq \psi\left(p(x, x^*)\right),
\]
which is a contradiction. Hence $p(x, x^*) = 0$ and this completes the proof of the theorem.\[\blacksquare\]

The next result is an immediate consequence of the Theorem 3.4 by taking $\psi(t) = 0$ for all $t \geq 0$.

**Corollary 3.5.** Let $A$ and $B$ be non-empty closed subsets of the metric space $(X, d)$ such that $A_0 \neq \emptyset$. Let $p$ be a $\tau$-distance on $X$ and $T: A \to B$ satisfies the following conditions:
(a) $T(A_0) \subseteq B_0$ and $(A, B)$ has the has the weak $P$-property with respect to $p$.
(b) $T$ is a continuous function on $A$ such that
\[
p(Tx, Ty) \leq p(x, y) - \phi(p(x, y)), \quad \forall x, y \in A
\]
where $\phi: [0, \infty) \to [0, \infty)$ is non-decreasing function also $\phi(t) = 0$ if and only if $t = 0$. Then $T$ has a best proximity point in $A$. Moreover, if $d(x, Tx) = d(x^*, T^x) = d(A, B)$ for some $x, x^* \in A$, then $p(x, x^*) = 0$.

The following result is the special case of the Corollary 3.5, obtained by setting $p = d$.

**Corollary 3.6.**[9] Let $(A, B)$ be a pair of two nonempty, closed subsets of a complete metric space $X$ such that $A_0$ is non-empty. Let $T: A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair $(A, B)$ has the $P$-property. Then there exists a unique $x^*$ in $A$ such that $d(x^*, T^x) = d(A, B)$.

**References**


