On a Pseudo Projective $\phi$ – Recurrent Sasakian Manifolds

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Abstract
The object of the present paper is to study the pseudo projective $\phi$ – recurrent Sasakian manifolds.

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1. Introduction
The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. As a weaker version of local symmetry, in 1977, Takahashi [9] introduced the notion of locally $\phi$ – symmetric Sasakian manifold and obtained their several interesting results. The properties of pseudo projective curvature tensor is studied by many geometers [17], [18], [19], [22] and obtained their some interesting results.

In this paper we shown that pseudo projective $\phi$ – recurrent Sasakian manifold is an Einstein manifold and in a pseudo projective $\phi$ – recurrent Sasakian manifold, the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1 – form $A$ are co-directional. Finally, we proved that a three dimensional locally pseudo – projective $\phi$ – recurrent Sasakian manifold is of constant curvature.

2. Preliminaries
Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is the structure vector field, $\eta$ is a 1 – form and $g$ is the Riemannian metric. It is well known that the structure $(\phi, \xi, \eta, g)$ satisfy

$$\phi^2 X = -X + \eta(X)\xi,$$

(a) $\eta(\xi) = 1,$ (b) $g(X, \xi) = \eta(X),$ (c) $\eta(\phi X) = 0,$ (d) $\phi \xi = 0,$

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\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]  
\[ (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \]  
\[ \nabla_X \xi = -\phi X, \]  
\[ (\nabla_X \eta)(Y) = g(X, \phi Y), \]

for all vector fields \(X, Y, Z\), where \(\nabla\) denotes the operator of covariant differentiation with respect to \(g\), then \(M^{2n+1}(\phi, \xi, \eta, g)\) is called a Sasakian manifold [1].

Sasakian manifolds have been studied by many authors such as De, Shaikh and Biswas [3], Takahashi [9], Tanno [15] and many others.

In a Sasakian manifold the following relations hold: [1]

\[ \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \]  
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \]  
\[ S(X, \xi) = 2n\eta(X), \]  
\[ S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \]

for all vector fields \(X, Y, Z\), where \(S\) is the Ricci tensor of type \((0,2)\) and \(R\) is the Riemannian curvature tensor of the manifold.

A Sasakian manifold is said to be an Einstein manifold if the Ricci tensor \(S\) is of the form

\[ S(X, Y) = \lambda g(X, Y), \]

where \(\lambda\) is a constant.

**Definition 2.1.** A Sasakian manifold is said to be a locally \(\phi -\) symmetric manifold if [9]

\[ \phi^2((\nabla_W R)(X, Y)Z) = 0, \]  
for all vector fields \(X, Y, Z, W\) orthogonal to \(\xi\).

**Definition 2.2.** A Sasakian manifold is said to be a locally pseudo projective \(\phi -\) symmetric manifold if

\[ \phi^2((\nabla_W \tilde{P})(X, Y)Z) = 0, \]  
for all vector fields \(X, Y, Z, W\) orthogonal to \(\xi\).

**Definition 2.3.** A Sasakian manifold is said to be pseudo projective \(\phi -\) recurrent Sasakian manifold if there exists a non-zero 1-form \(A\) such that

\[ \phi^2((\nabla_W \tilde{P})(X, Y)Z) = A(W)\tilde{P}(X, Y)Z, \]  
for arbitrary vector fields \(X, Y, Z, W\), where \(\tilde{P}\) is a pseudo projective curvature tensor given by [17]

\[ \tilde{P}(X, Y)Z = \frac{r}{2^n+1} \left[ \frac{\alpha}{2^n} + b \right] [g(Y, Z)X - g(X, Z)Y]. \]

where \(a\) and \(b\) are constants such that \(a, b \neq 0\). If \(a = 1\) and \(b = -\frac{1}{2^n}\) Then (14) takes of the form

\[ \tilde{P}(X, Y)Z = R(X, Y)Z - \frac{1}{2^n} [S(Y, Z)X - S(X, Z)Y] = P(X, Y)Z. \]

where \(P\) is the projective curvature tensor [21]. Hence the Projective curvature \(P\) is a particular case of the tensor \(\tilde{P}\). For the reason \(\tilde{P}\) is called Pseudo projective curvature tensor, where \(R\) is the Riemann curvature tensor \(S\) is Ricci tensor and \(r\) is the scalar curvature.

If the 1-form \(A\) vanishes, then the manifold reduces to a locally pseudo projective \(\phi -\) symmetric manifold.
3. Pseudo projective $\phi$ – recurrent Sasakian manifold

In this section we consider a Sasakian manifold which is pseudo projective $\phi$ – recurrent Sasakian manifold. Then by virtue of (1) and (3), we get

$$-(\nabla_W \tilde{P}) (X,Y)Z + \eta \left( (\nabla_w \tilde{P}) (X,Y)Z \right) \xi = A(W) \tilde{P}(X,Y)Z,$$

from which it follows that

$$-g \left( (\nabla_W \tilde{P}) (X,Y)Z, U \right) + \eta \left( (\nabla_w \tilde{P}) (X,Y)Z \right) \eta (U) = A(W) g(\tilde{P}(X,Y)Z, U).$$

Let $\{e_i\}, i = 1, 2, \ldots, 2n + 1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (16) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$(\nabla_W S)(Y,Z) = A(W) \left[ S(Y,Z) - \left( \frac{r}{2n+1} \right) g(Y,Z) \right].$$

Replacing $Z$ by $\xi$ in (17) and using (2) and (9), we get

$$(\nabla_W S)(Y,\xi) = A(W) \left[ 2n - \left( \frac{r}{2n+1} \right) \right] \eta(Y).$$

Now we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi),$$

using (5), (6) and (9) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi) = 2ng(\phi Y, W) + S(Y, \phi W).$$

In view of (18) and (19), we get

$$S(Y, \phi W) = -2ng(\phi Y, W) + A(W) \left[ 2n - \left( \frac{r}{2n+1} \right) \right] \eta(Y).$$

Replacing $Y$ by $\phi Y$ and using (3), (4) and (10) in (20), we get

$$S(Y, W) = 2ng(Y, W).$$

for all $Y, W$.

Hence, we can state the following theorem:

**Theorem 3.1** A pseudo projective $\phi$ – recurrent Sasakian manifold $(M^{2n+1}, g)$ is an Einstein manifold.

Now from (15), we have

$$(\nabla_W \tilde{P})(X,Y)Z = \eta \left( (\nabla_w \tilde{P}) (X,Y)Z \right) \xi - A(W) \tilde{P}(X,Y)Z.$$

Using (14) in (22), we get

$$a(\nabla_W R)(X,Y)Z = a\eta \left( (\nabla_w R)(X,Y)Z \right) \xi - aA(W)R(X,Y)Z$$

$$+ b \left[ (\nabla_W S)(Y,Z) \eta(X) - (\nabla_W S)(X,Z) \eta(Y) \right] \xi$$

$$- b \left[ (\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \right]$$

$$- bA(W)[S(Y,Z)X - S(X,Z)Y]$$

$$+ \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(W) [g(Y,Z)X - g(X,Z)Y].$$

From (23) and the Bianchi identity, we get

$$aA(W) \eta(R(X,Y)Z) + aA(X) \eta(R(Y,W)Z) + aA(Y) \eta(R(W,X)Z)$$

$$= bA(W)[S(X,Z) \eta(Y) - S(Y,Z) \eta(X)]$$

$$- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(W) [g(Y,Z) \eta(Y) - g(Y,Z) \eta(X)]$$

$$+ bA(X)[S(Y,Z) \eta(W) - S(W,Z) \eta(Y)]$$

$$- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(X) [g(Y,Z) \eta(W) - g(W,Z) \eta(Y)]$$

$$+ bA(Y)[S(W,Z) \eta(X) - S(X,Z) \eta(W)]$$

$$- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(Y) [g(W,Z) \eta(X) - g(X,Z) \eta(W)].$$

By virtue of (8), we obtain from (24) that
\[ aA(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + aA(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] + aA(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)] \\
= bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)] \\
- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \\
+ bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)] \\
- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] \\
+ bA(Y)[S(W, Z)\eta(X) - S(X, Z)\eta(W)] \\
- \left( \frac{r}{2n+1} \right) \left[ \frac{a}{2n} + b \right] A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)] \\
\]

(25)

Putting \( Y = Z = e_1 \) in (25) and taking summation over \( i, 1 \leq i \leq 2n + 1 \) we get

\[ A(W)\eta(X) = A(X)\eta(W), \]

(26)

for all vector fields \( X, W \). Replacing \( X \) by \( \xi \) in (26), we get

\[ A(W) = \eta(W)\eta(\rho), \]

(27)

for any vector field \( W \), where \( A(\xi) = g(\xi, \rho) = \eta(\rho) \), \( \rho \) being the vector field associated to the \( 1 \) - form \( X \) i.e., \( A(X) = g(X, \rho) \). From (27),

we can state the following theorem:

**Theorem 3.2** In a Pseudo projective \( \phi \) - Sasakian manifold \( (M^{2n+1}, g)(n \geq 1) \), the characteristic vector field \( \xi \) and the vector field \( \rho \) associated to the \( 1 \) - form \( A \) are co-directional and the \( 1 \) - form \( A \) is given by (27).

4. **On a 3 - dimensional Locally Pseudo Projective \( \phi \) - Recurrent Sasakian Manifold**

On a 3 - dimensional Sasakian Manifold Ricci tensor and curvature tensor has the following form

\[ S(X, Y) = \left( \frac{r}{2} - 1 \right) g(X, Y) - \left( \frac{r}{2} - 3 \right) \eta(X)\eta(Y) \]

(28)

\[ R(X, Y)Z = \left( \frac{r-4}{2} \right) [g(Y, Z)X - g(X, Z)Y] \\
- \left( \frac{r-6}{2} \right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \]

(29)

Taking covariant differentiation of (29), we get

\[ (\nabla_W R)(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y - g(Y, Z)\eta(X)\xi \\
+ g(X, Z)\eta(Y)\xi - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] \\
- \left( \frac{r-6}{2} \right) [g(Y, Z)(\nabla_W \eta)(X)\xi + g(Y, Z)\eta(X)(\nabla_W \xi) \\
- g(X, Z)(\nabla_W \eta)(Y)\xi + g(X, Z)\eta(Y)(\nabla_W \xi) ] \\
+ (\nabla_W \eta)(Y)\eta(Z)X + (\nabla_W \eta)(Z)\eta(Y)X \\
- (\nabla_W \eta)(X)\eta(Z)Y - (\nabla_W \eta)(Z)\eta(X)Y. \]

(30)

Taking \( X, Y, Z, W \) orthogonal to \( \xi \) and using (5) and (6), we get

\[ (\nabla_W R)(X, Y)Z = \frac{dr(W)}{2} [g(Y, Z)X - g(X, Z)Y] \\
- \left( \frac{r-6}{2} \right) [g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)]\xi \]

(31)

from (31) it follows that
\[
\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2} [g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y]
\] (32)

Now, taking \(X, Y, Z, W\) orthogonal to \(\xi\) and using (1) and (2) in (32), we get

\[
\phi^2(\nabla_W R)(X,Y)Z = -\frac{dr(W)}{2} [g(Y,Z)X - g(X,Z)Y]
\] (33)

Differentiating covariantly (14) with respect to \(W\) (for \(n = 1\), we get

\[
(\nabla_W \bar{\nabla})(X,Y)Z = a(\nabla_W R)(X,Y)Z + b(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y
\]

\[-\frac{dr(W)}{3} [a + b] g(Y,Z)X - g(X,Z)Y].
\] (34)

Using (28) in (34) and then taking \(X, Y, Z, W\) orthogonal to \(\xi\), we get

\[
(\nabla_W \bar{\nabla})(X,Y)Z = a(\nabla_W R)(X,Y)Z - \frac{dr(W)}{6} [a - b] g(Y,Z)X - g(X,Z)Y.
\] (35)

Now, applying \(\phi^2\) to the both side of (35), we get

\[
\phi^2(\nabla_W \bar{\nabla})(X,Y)Z = a\phi^2(\nabla_W R)(X,Y)Z - [a - b] \frac{dr(W)}{6} [g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
\] (36)

Using (13), (33), (1) in (36), we obtain

\[
A(W)\bar{\nabla}(X,Y)Z = -a \frac{dr(W)}{2} [g(Y,Z)X - g(X,Z)Y]
\]

\[+ [a - b] \frac{dr(W)}{6} [g(Y,Z)X - g(X,Z)Y + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi].
\] (37)

taking \(X, Y, Z\) orthogonal to \(W\), we get

\[
\bar{\nabla}(X,Y)Z = -\frac{(2a+b)dr(W)}{6A(W)} [g(Y,Z)X - g(X,Z)Y]
\] (38)

Putting \(W = \{e_i\}\) in (38), where \(\{e_i\}, i = 1,2,3\) is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \(i\), \(1 \leq i \leq 3\), we obtain

\[
\bar{\nabla}(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y].
\]

where \(\lambda = \left[-\frac{(2a+b)dr(e_i)}{6A(e_i)}\right]\) is a scalar, since \(A\) is a non-zero \(1\) - form. Then, by Schur’s theorem \(\lambda\) will be a constant on the manifold. Hence, we can state the following theorem:

**Theorem 4.1** On a 3 - dimensional locally Pseudo-projective \(\phi\) - recurrent Sasakian manifold, pseudo-projective curvature tensor is of the form of constant curvature.

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**References**


