Multiple solution to \((p,q)\)-Laplacian systems with concave nonlinearities

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Abstract

In this paper we study the \((p,q)\)-Laplacian systems with concave nonlinearities. Using some asymptotic behavior \(f\) at zero and infinity, three nontrivial solutions are established.

Keywords: Nonlinear boundary value problem, Concave nonlinearity, \((p,q)\)-Laplacian systems, Variational method, Multiple solutions.

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1 Introduction

In this paper, we consider problems

\[
\begin{cases}
  -\Delta_p u = \lambda |u|^{p-2}u + f_u(x,u,v) & \text{in } \Omega \\
  -\Delta_q v = \lambda |v|^{q-2}v + f_v(x,u,v) & \text{in } \Omega \\
  u = v = 0 & \text{in } \partial \Omega
\end{cases}
\]  

(1.1)

Where \( \Omega \subset \mathbb{R}^N, (N \geq 1) \) is a bounded with smooth domain and \( F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R}) \).

The functional corresponding to problems (1.1) is

\[
J_\lambda(u,v) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx - \lambda \frac{1}{p} \int_\Omega |u|^p \, dx - \lambda \frac{1}{q} \int_\Omega |v|^q \, dx - \int_\Omega F(x,u,v) \, dx
\]

Let \( W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \) with the norm
\[ \| (u, v) \| = \| \nabla u \|_p + \| \nabla v \|_q = \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla v|^q \, dx \right)^{\frac{1}{q}}. \]

It is well known operator -\( \Delta \) has a sequence of eigenvalues \( \{ \lambda_k \} \) satisfying

\[ 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty. \]

For general \((p, q) \in (1, +\infty), (-\Delta_p, -\Delta_q) \) has a smallest eigenvalue, i.e., the principle value, \( \lambda_1 \), which is positive, isolated, simple (see \[\text{[2]}\]) and admit the following variational characterization

\[ \lambda_1 = \inf_{0 \neq u, v \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |\nabla v|^q \, dx}{\int_{\Omega} |u|^p \, dx + \int_{\Omega} |v|^q \, dx} \quad (1.2) \]

Furthermore, the \( \lambda_1 \) - eigenfunctions do not change in \( \Omega \), and by the maximum principle we may suppose that \( \phi_1 > 0 \) is a \( \lambda_1 \) - eigenfunction. There are many paper concerned with the resonance problem. In \[\text{[7]}\] L. Shi proved that there exists \( \lambda^* > 0 \) such that p-Laplacian multiple solutions for a class of \((p,q)\)-Laplacian systems \((1.1)\). Consider the following conditions hold:

(i) \( f(0,0) = 0 \).
(ii) \( f \in C^1(\Omega \times R^2, R) \) and \( f'(0,0) > \lambda_1 \).
(iii) For some positive integer \( k \geq 1 \),

\[ \lim_{|(s,t)| \to -\infty} \frac{f(x,s,t)}{|(s,t)|} < \lambda_1 \leq \lambda_k < \lim_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|(s,t)|}. \]

In this paper we extend this result to the case \( 1 < p, q < +\infty \); Furthermore, here the \( \lim_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|(s,t)|} \) relaxed to

\[ \mu_2 \leq \liminf_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|(s,t)|} \leq \limsup_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|(s,t)|} \leq \mu_3, \]

Where \( \mu_2, \mu_3 \in (\lambda_1, +\infty) \).

Our main result is as follows.

**Theorem 1.1.** Assume that \( f \in (\overline{\Omega} \times R^2, R) \) and \( f(x,0,0) = 0 \) a.e. If the following conditions hold

(i) \( \text{There exists constant } \mu_0 > \lambda_1 \text{ such that} \)

\[ \liminf_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|s|^{p-2}s + |t|^{q-2}t} \geq \mu_0 \quad \text{Uniformly for } a.e. x \in \Omega; \quad (1.3) \]

(ii) \( \text{There exist constants } \mu_1, \mu_2, \mu_3 > 0 \text{ with } \mu_1 < \lambda_1 < \mu_2 \text{ such that} \)

\[ \limsup_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|s|^{p-2}s + |t|^{q-2}t} \leq \mu_1, \quad (1.4) \]
\[ \mu_2 \leq \lim_{(s, t) \to +\infty} \frac{f(x, s, t)}{|s|^2s + |t|^q t} \leq \lim_{(s, t) \to +\infty} \frac{\sup f(x, s, t)}{|s|^p |s^2s + |t|^q t} \leq \mu_3. \]

Hold uniformly for a.e. \( x \in \Omega \), then there exist such that problem (1.1) admits at least three nontrivial solutions for \( \lambda \in (0, \lambda^*) \).

2 Proof of the main result

Define the functional \( J_\lambda : W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R} \) by

\[ J_\lambda(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx - \frac{\lambda}{p} \int_\Omega |u|^p \, dx - \frac{\lambda}{q} \int_\Omega |v|^q \, dx - \int_\Omega F(x, u, v) \, dx. \]

Clearly, \( J_\lambda \in C^1(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \mathbb{R}) \). It is obvious that the critical points of correspond to the weak solutions of problem (1.1).

Lemma 2.1. Assume that the assumptions of theorem 1.1 hold. Then the functional \( J_\lambda(u, v) \) satisfies the (PS) condition.

**Proof:** Assume that \( \{(u_n, v_n) = z_n\}_{n \in \mathbb{N}} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \) is a (PS) sequence, i.e., for some \( M > 0 \),

\[ |J_\lambda(u_n, v_n)| \leq M, \quad \forall J_\lambda(u_n, v_n) \to 0 \text{ as } n \to \infty. \quad (2.1) \]

It suffices to prove that \( \{(u_n, v_n) = z_n\} \) is bounded in \( W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \). In fact, if not, we may assume by contradiction that there exist a sequence of \( \{(u_n, v_n) = z_n\} \) with \( \|(u_n, v_n)\| \to +\infty \) and \( \{\varepsilon\} \) with \( \varepsilon_n \to 0 \) in

\[ W_0^{-1,p'}(\Omega) \times W_0^{-1,q'}(\Omega) \]

such that

\[ -\Delta_p u_n = -\lambda |u_n|^{p-2}u + f_u(x, u_n, v_n) \text{ in } W_0^{-1,p'}(\Omega) \quad (2.2) \]

Taking \(-u_n^-\) as test function in (2.2), we obtain

\[ \|\nabla u_n^-\|^p = \int_\Omega \lambda |u_n^-|^p \, dx - \int_\Omega f_u(x, u_n, v_n)u_n^- \, dx - \int_\Omega \varepsilon_n u_n^- \, dx. \]

Similarly,

\[ \|\nabla v_n^-\|^q = \int_\Omega \lambda |v_n^-|^q \, dx - \int_\Omega f_v(x, u_n, v_n)v_n^- \, dx - \int_\Omega \varepsilon_n v_n^- \, dx. \]
In view of (4.1), for any \( \varepsilon \in (0, \lambda_1 - \mu_1) \), there exists \( C = C(\varepsilon) > 0 \) such that

\[
f(x, s, t) \geq (\mu_1 + \varepsilon)(|s|^{p-2}s + |t|^{q-2}t) - C \quad \forall s, t < 0 \quad a.e., x \in \Omega.
\]  

(2.3)

Then by the Sobolev embedding and Poincare inequality there exist \( C_1, C_2 > 0 \) such that

\[
\|\nabla u_n\|_p^p + \|\nabla v_n\|_q^q \leq 
\]

\[
\int_{\Omega} \lambda(|u_n|^p + |v_n|^q)dx + \int_{\Omega} (\mu_1 + \varepsilon)(|u_n|^p + |v_n|^q)dx - \int_{\Omega} (C - \varepsilon_n) (u_n + v_n)dx 
\]

\[
\leq C_1 \int_{\Omega} \lambda (\|u_n\|^p + \|v_n\|^q)dx + \int_{\Omega} \frac{\mu_1 + \varepsilon}{\lambda_1} (\|u_n\|^p + \|v_n\|^q)dx 
\]

\[
+ C_2 \int (\|u_n\| + \|v_n\|) \ dx.
\]

Hence by \( \mu_1 + \varepsilon < \lambda_1 \), it follows that \( \{(u_n, v_n) = z_n^{-}\} \subset W_0^{1,p} (\Omega) \times W_0^{1,q} (\Omega) \) is bounded. For any \( n \), we take \( \psi_{n,k} = -((u_n + v_n)k)^- \) with \( k > 0 \) as test function in (2.2), using again (2.3), we get

\[
\int_{\Omega} |\nabla \psi_{n,k}|^p dx \leq 
\]

\[
- \int_{\Omega} \lambda(|u_n|^{p-2} + |v_n|^{q-2}) \psi_{n,k} dx + \int_{\Omega} (\mu_1 + \varepsilon) ((|u_n|^{p-2} + |v_n|^{q-2}) \psi_{n,k} dx 
\]

\[
- \int_{\Omega} (C - \varepsilon_n) \psi_{n,k} \ dx.
\]

We can obtain that \( \{\|(u_n, v_n)\|_{\infty}\} \) is bounded. By the standard regularity theory (see [4]), it follow that there exists \( C_3 > 0 \) such that, for every \( n, u_n \in C^{1,\sigma} (\overline{\Omega}) \) for some \( \sigma > 0 \) and

\[
\|(\nabla u_n, \nabla v_n)\|_{\infty} \leq C_3 (1 + \|(u_n + v_n)\|_{\infty}).
\]  

(2.4)

Thus by \( \|(u_n, v_n)\|_{\infty} \to +\infty \) it follows that \( \|(u_n^+, v_n^+)\|_{\infty} \to +\infty. \)  

(2.5)

We may assume that \( z_n = \|(u_n, v_n)\| \to \infty \) as \( n \to \infty \). Define \( \hat{u}_n = \frac{u_n}{z_n}, \hat{v}_n = \frac{v_n}{z_n} \).
Denote \( g(x,s,t) = -\lambda (|s|^{p-2}s + |t|^{q-2}t) + f(x,s,t) \). In view of (2.1), for all \( \phi \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \), we have

\[
\int_\Omega \left| \nabla \hat{u}_n \right|^{p-2} \nabla \hat{u}_n \nabla \phi - \frac{\partial u(x,u_n,v_n)}{|x|^{p-1}} \right] dt \to 0 \tag{2.6}
\]

Since \( g \) is countinous and \( \| (u_n,v_n^-) \|_\infty \) is uniformly bounded, using (1.3)-(1.5) and (2.5), there exist constant \( C_4, C_5 > 0 \) and \( \varepsilon \in (0, \mu_2 - \lambda_1) \) such that

\[
(\mu_2 - \varepsilon) |\hat{u}_n(x)|^{p-1} - \frac{C_4}{\| z_n \|^{p-1}_\infty} \leq g(x,u_n,v_n) \leq (\mu_3 + \varepsilon) |\hat{u}_n|^{p-1} + \frac{C_5}{\| z_n \|^{p-1}_\infty}
\]

Hold uniformly for \( a.e. x \in \Omega \). In a similarly way, we get \( \hat{u}_n \to \hat{u} \). By the regularity theory (see [4]), there exists a constant \( M_2 > 0 \) such that, for every \( n \), \( \| (\hat{u}_n, \hat{v}_n) \|_{C^{1,\sigma}} \leq M_2 \), set \( w_n = \frac{z_n}{\| z_n \|_\infty} \). Then by the compact imbedding of \( C^{1,\sigma}(\Omega) \) into \( C^1(\Omega) \), passing to a subsequence if possible, we have

\[
w_n \to w_0 \quad \text{in} \quad C^1(\Omega) \tag{2.8}
\]

With \( \| w_0 \|_\infty = 1 \), then \( (\hat{u}_n, \hat{v}_n) \) is bounded Which \( \| \hat{u}_n \|_{1,p} + \| \hat{v}_n \|_{1,q} = 1 \).

Using again that \( \| (\hat{u}_n, \hat{v}_n) \| \) is bounded and \( \hat{u}_n = \frac{u_n^+ - u_n^-}{\| z_n \|_\infty} \), we can see that \( \hat{u}_0 \geq 0 \) and \( \hat{u}_0 \not\equiv 0 \), similarly for \( \hat{v}_n = \frac{v_n^+ - v_n^-}{\| z_n \|_\infty} \) and we can see that \( \hat{v}_0 \geq 0 \) and \( \hat{v}_0 \not\equiv 0 \).

Denote \( \alpha_n(x) = \frac{g(x,u_n,v_n)}{\| z_n \|_\infty^{p-1}} \). By (2.7) and (2.8) if follows that there exists \( \alpha \in L^\infty(\Omega) \) satisfying

\[
\mu_2 - \varepsilon \leq \alpha(x) \leq \mu_3 + \varepsilon \tag{2.9}
\]

Such that

\[
\alpha_n \to \alpha (|\hat{u}_0|^{p-2}u_0 + |\hat{v}_0|^{q-2}v_0) \quad \text{weakly in} \quad L^\infty(\Omega) \tag{2.10}
\]

By (2.6), (2.8), (2.10) we obtain

\[
\int_\Omega \left| \nabla \hat{u}_0 \right|^{p-2} \nabla u_0 \nabla \phi \, dx = \int_\Omega \left[ \alpha(x) |\hat{u}_0|^{q-2}u_0 \right] \phi \, dx
\]

For every \( \phi \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \). Consequently, similarly

\[
\int_\Omega \left| \nabla \hat{v}_0 \right|^{p-2} \nabla v_0 \nabla \phi \, dx = \int_\Omega \left[ \alpha(x) |\hat{v}_0|^{q-2}v_0 \right] \phi \, dx
\]
\((\bar{u}_0, \bar{v}_0)\) is a nontrivial solution of
\[
\begin{align*}
-\Delta_p w_0 &= \alpha(x) w_0^{p-1} & \text{in } \Omega \\
-\Delta_q w_0 &= \alpha(x) w_0^{q-1} & \text{in } \Omega \\
 w_0 &= 0 & \text{on } \partial \Omega
\end{align*}
\tag{2.11}
\]

By the maximum principle of Vazquez's [9], it follows that \(w_0(x) > 0\) for \(x \in \Omega\).

Furthermore, there is a positive constant \(\delta > 0\) and \(\varphi = (\varphi_1, \varphi_2)\) such that
\[
\delta \varphi \leq w_0 \quad \text{on } \partial \Omega
\tag{2.12}
\]

By (2.9), (2.11) and \(\mu_2 > \lambda_1\), for any \(\varepsilon \in (0, \frac{\mu_2 - \lambda_1}{2})\), we get
\[
-\Delta_p w_0 > (\lambda_1, \lambda_1 + \varepsilon) w_0^{p-1}
\tag{2.13}
\]

And
\[
-\Delta_q w_0 > (\lambda_1, \lambda_1 + \varepsilon) w_0^{q-1}.
\]

Take \(\psi = (\psi_1, \psi_2)\) and \(\varepsilon \delta \varphi\) and \(\mu \in (\lambda_1, \lambda_1 + \varepsilon)\). Then we have
\[
-\Delta_p \psi = \lambda_1 \psi^{p-1} \leq \mu \psi^{p-1}
\]
and
\[
-\Delta_q \psi = \lambda_1 \psi^{q-1} \leq \mu \psi^{q-1}.
\]

By (2.12) and (2.13), by the method of sub and supersolution, for any \(\varepsilon > 0\) small enough, we can obtain a solution \((\bar{u}, \bar{v}) \in [\psi, w_0]\) of the following problems
\[
\begin{align*}
-\Delta_p u &= \mu w^{p-1} & \text{in } \Omega \\
-\Delta_q v &= \mu w^{q-1} & \text{in } \Omega \\
u = v &= 0 & \text{on } \partial \Omega
\end{align*}
\]

However, this is contrary to this fact that \(\lambda_1\) is isolated. Hence \(\{(u_n^+, v_n^+)\}\) is also uniformly bounded. Thus by (2.4) we can see that the sequence \(\{(u_n, v_n)\}\) is uniformly bounded. Then using standard arguments we can see that \(J_{\lambda}\) satisfies the (PS) condition. This completes the proof.

Define
Define the corresponding functional $J^\pm_\lambda: W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \to \mathbb{R}$ as follows.

$$J^+_{\lambda}(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx - \frac{\lambda}{p} \int_\Omega |u^+|^p \, dx - \frac{\lambda}{q} \int_\Omega |v^+|^q \, dx - \int F_+(x, u, v) \, dx,$$

$$J^-_{\lambda}(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{1}{q} \int_\Omega |\nabla v|^q \, dx - \frac{\lambda}{p} \int_\Omega |u^-|^p \, dx - \frac{\lambda}{q} \int_\Omega |v^-|^q \, dx - \int F_-(x, u, v) \, dx,$$

where $\nabla F = (f_u, f_v)$. Obviously, $J^+_{\lambda} \in C^1(W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega), \mathbb{R})$. Similarly, define $f^-(x, s, t) = \begin{cases} f(x, s, t) & t, s \leq 0 \\ 0 & t, s \geq 0 \end{cases}$

Define the corresponding functional $J^-_{\lambda(u, v)}: W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \to \mathbb{R}$ as follows.

Using similar arguments as in the proof of lemma 2.1, we obtain the following result.

**Lemma 2.2.** The functional $J^\pm_{\lambda}$ satisfies the (PS) condition.

To prove Theorem 1.1, we prove some preliminary results as follows.

**Lemma 2.3.** If $(u^\pm, v^\pm)$ is a local minimizer of $J^\pm_{\lambda}$, then it is also a local minimizer of $J_{\lambda}$.

**Proof:** By Theorem 1.1 of Garcia Azorero, Peral Alonso and Manfredi[5], we just need to show that $(u^\pm, v^\pm)$ is a local minimizer of $J_{\lambda}$ in the $C^1$ topology. By the assumption it follow that $(u^\pm, v^\pm)$ is a $C^1_0(\overline{\Omega})$-local minizmer of $J^\pm_{\lambda}$ i.e., there exists $\rho_1 > 0$ such that

$$J^\pm_{\lambda}(u^\pm, v^\pm) \leq J^\pm_{\lambda}(u, v), \quad \forall u \in B_{\rho_1}(u^\pm, v^\pm)$$

where $B_{\rho_1}(u^\pm, v^\pm) = \{(u, v) \in C^1_0(\overline{\Omega}) : \|(u, v) - (u^\pm, v^\pm)\|_{C^1} < \rho_1 \}$. By (1.4), (1.5), we can see that $f$ is of p-linear growth [5]. Then, for $(u, v) \in B_{\rho_1}(u^\pm, v^\pm)$, we obtain
\[
J_\lambda (u, v) - J_\lambda (u^\pm, v^\pm) = J_\lambda (u, v) - J_\lambda^+ (u^\pm, v^\pm)
\]
\[
\geq \frac{\lambda}{p} \int_{\Omega} [|(u, v)|^p - |(u^\pm, v^\pm)|^p] \, dx
\]
\[
+ \frac{\lambda}{q} \int_{\Omega} [|(u, v)|^q - |(u^\pm, v^\pm)|^q] \, dx - \int_{\Omega} [F(x, u, v) - F(\pm u, \pm v)] \, dx
\]
\[
= \frac{\lambda}{p} \int_{\Omega} |(u^\pm, v^\pm)|^p \, dx + \frac{\lambda}{q} \int_{\Omega} |(u^\pm, v^\pm)|^q \, dx - \int_{\Omega} F_\pm (x, u, v) \, dx
\]
\[
\geq \frac{\lambda}{p} \int_{\Omega} |(u^\pm, v^\pm)|^p \, dx + \frac{\lambda}{q} \int_{\Omega} |(u^\pm, v^\pm)|^q \, dx
\]
\[
- C \left( \int_{\Omega} |(u^\pm, v^\pm)|^p \, dx + \int_{\Omega} |(u^\pm, v^\pm)|^q \, dx \right)
\]
\[
\geq \left[ \frac{\lambda}{q} - C \| (u^\pm, v^\pm) \|_{\infty}^{p-q} \right] \int_{\Omega} |(u^\pm, v^\pm)|^q \, dx
\]
\[
+ \left[ \frac{\lambda}{p} - C \| (u^\pm, v^\pm) \|_{\infty}^{q-p} \right] \int_{\Omega} |(u^\pm, v^\pm)|^p \, dx
\]

Note \( \rho_1 \to 0 \) implies \( \| (u^-, v^-) \|_{\infty} \to 0 \), together with \( 1 < p, q < +\infty \), we can see that there exists \( \rho_2 > 0 \) small enough such that

\[ J_\lambda (u^\pm, v^\pm) \leq J_\lambda (u, v), \quad \forall (u, v) \in B_{\rho_2} (u^\pm, v^\pm), \]

Where \( B_{\rho_2} (u^\pm, v^\pm) = \{ (u, v) \in C^1_0 (\overline{\Omega}) : \| (u, v) - (u^\pm, v^\pm) \|_{C^1} < \rho_2 \} \). This completes the proof.

**Lemma 2.4.** 0 is a local minimize of \( J_\lambda^+ \) and \( J_\lambda \) for \( \lambda > 0 \).

**Proof:** we just consider the case of \( J_\lambda \). The other cases can be treated similarly. As shown in the proof of lemma 2.3, it suffices to prove that 0 is a local minimizer of \( J_\lambda \) in the \( C^1 \) topology. In fact, for \( (u, v) \in C^1_0 (\overline{\Omega}) \), we have
If we define $B_{p_3}(0,0) = \{(u,v) \in C_0^1(\Omega) : \|u,v\|_{C^1} < p_3\}$, where $p_3 \in (0, \left(\frac{\lambda}{c_q}\right)^{\frac{1}{p-q}}, \left(\frac{\lambda}{c_p}\right)^{\frac{1}{p-q}})$, then it follows that

$$J_\lambda(u,v) \geq 0, \quad \forall (u,v) \in B_{p_3}(0,0).$$

The proof is complete.

**Lemma 2.5.** There exist $\lambda^*, t_1, t_2 > 0$ such that, for $\lambda \in (0, \lambda^*)$

$$J_\lambda(\pm t_1 \phi_1, \pm t_2 \phi_2) < 0 \quad (2.14)$$

**Proof:** By (1.3)-(1.5), for any given $\varepsilon > 0$ and $r \in \left(p, \frac{p_n}{n-p}\right)$ if $n > p$; $r \in (p, +\infty)$

If $1 \leq n \leq p$, there exist $C > 0$ such that

$$|pF(x,z) - \mu_3|z|^p| \leq \varepsilon|z|^p + C|z|^r.$$  

Then, taking $\varepsilon < \mu_3 - \lambda_1$, we have

$$J_\lambda(t_1 \phi_1, t_2 \phi_2) = \frac{|t_1|^p}{p} \|\phi_1\|^p + \frac{|t_2|^q}{q} \|\phi_2\|^q + \frac{|t_1|^q}{q} \lambda \int_\Omega \phi_1^q dx + \frac{|t_2|^q}{q} \lambda \int_\Omega \phi_2^q dx - \int_\Omega F(t_1 \phi_1, t_2 \phi_2) dx \leq \frac{|t_1|^p}{p} \|\phi_1\|^p + \frac{|t_2|^q}{q} \|\phi_2\|^q + \frac{|t_2|^q}{q} \lambda \int_\Omega \phi_2^q dx - \frac{|t_1|^p}{p} \mu_3 \int_\Omega \phi_1^p dx + \frac{|t_1|^p}{p} F(t_1 \phi_1, t_2 \phi_2) dx + \frac{|t_1|^p}{p} C \int_\Omega \phi_1^r dx = [\lambda_1 - \mu_3 + \varepsilon] \frac{|t_1|^p}{p} \int_\Omega \phi_1^p dx + \frac{|t_2|^q}{q} \lambda \int_\Omega \phi_2^q dx + \frac{|t_2|^q}{q} \lambda \int_\Omega \phi_2^q dx + \frac{|t_1|^p}{p} C \int_\Omega \phi_1^r dx \leq -\left(\frac{\mu_3 - \varepsilon}{\lambda_1} - 1\right) \frac{|t_1|^p}{p} \|\phi_1\|^p + C(\lambda |t_1|^q + |t_1|^r)^p)\|\phi_1\|^p$$

Define $\varphi(z) = \lambda z^{q-p} + z^{r-p}$ for $z \geq 0$, where $\delta \equiv \frac{1}{p} \left(\frac{\mu_3 - \varepsilon}{\lambda_1} - 1\right) > 0$.

Then $\varphi'(z) = \lambda(q-p)z^{q-p-1} + (r-p-1)z^{r-p-1}$.  


It is easily seen that \( \phi'(z_0) = 0 \) if \( z_0 = \left( \frac{\lambda(p-q)}{r-p-1} \right)^{\frac{1}{r-q}} \). Then we have
\[
\phi(z_0) = \left[ \delta_0^{\frac{q-p}{r-q}} + \delta_0^{\frac{r-p}{r-q}} \right]^{\frac{r-p}{r-q}} \lambda^{\frac{r-p}{r-q}}.
\]
Hence if taking \( |t| = z_0 \), there exists \( \lambda^* > 0 \) such that if \( \lambda < \lambda^* \) then
\[
C\phi(z_0) < \frac{1}{p} \left( \frac{\mu_3}{\lambda_1} - 1 \right).
\]
Thus we can see that (2.14) hold for \( \lambda \in (0, \lambda^*) \) if we take \( t_1 = z_0 \).

**Proof of theorem 1.1.** By lemma 2.4, 0 is a local minimizer of \( J_\lambda^\pm \) and \( J_\lambda \) with \( J_\lambda(0,0) = J_\lambda(0,0) = 0 \). In view of lemma 2.5, there exist \( t_1, t_2, \lambda^* > 0 \) such that, for
\[
\lambda \in (0, \lambda^*), \inf_{W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)} J_\lambda^\pm(u, v) \leq J_\lambda(\pm t_1 \phi_1, \pm t_2 \phi_2) < 0 \quad (2.15)
\]
Then \( J_\lambda^\pm \) has a nontrivial critical point \((u^\pm, v^\pm)\) of mountain pass type with \( J_\lambda^\pm(u^\pm, v^\pm) > 0 \), which implies that \((u^\pm, v^\pm)\) is a weak solution of the following \((p,q)\)-Laplacian
\[
\begin{align*}
-\Delta_p u &= \lambda |u^\pm|^{p-2} + f_u^\pm(x, u, v) \quad \text{in } \Omega \\
-\Delta_q v &= \lambda |v^\pm|^{q-2} + f_v^\pm(x, u, v) \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial\Omega
\end{align*}
\]
(2.16)
By the weak maximum principle we can see that \((\pm u_1^\pm, \pm v_1^\pm) \geq 0 \) in \( \Omega \), which implies that \((u_1^\pm, v_1^\pm)\) is also a solution of system (1.1) and
\[
J_\lambda(u^\pm, v^\pm) = J_\lambda^\pm(u^\pm, v^\pm).\]
In addition, by the fatter of (1.4) it follows that the functional \( J_\lambda^- \) is coercive on \( W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \) and hence bounded below. Combing with (2.15) implies that \( J_\lambda^- \) has a nontrivial global minimizer \((u_2^-, v_2^-) \) with \( J_\lambda^-(u_2^-, v_2^-) < 0 \). Then by lemma 2.3 we can see that \((u_2^-, v_2^-) \) is a local minimizer \( J_\lambda^- \). Thus (1.1) has at least nontrivial solutions \((u_1^-, v_1^-), (u_2^-, v_2^-), (u_1^+, v_1^+)\).

**References**


