On Lorentzian $\alpha$-Sasakian manifolds

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Abstract
We study Ricci-semi symmetric, $\phi$-Ricci semisymmetric and $\phi$-symmetric Lorentzian $\alpha$-Sasakian manifolds. Also, we study a Lorentzian $\alpha$-Sasakian manifold satisfies $S(X,\xi).R = 0$.

keywords: Ricci semisymmetric Lorentzian $\alpha$-Sasakian manifold, $\phi$-Ricci symmetric Lorentzian $\alpha$-Sasakian manifold, $\phi$-symmetric Lorentzian $\alpha$-Sasakian manifold.

1 Introduction
The notion of local symmetry of Riemannian manifolds have been weakened by many authors in several ways to the different extent. As a weaker version of local symmetry, Takahashi [6], introduced the notion of locally $\phi$-symmetry on sasakian manifolds. In respect of contact Geometry, the notion of $\phi$-symmetry was introduced and studied by Boeckx, Buecken and Vanhecke [2], with several examples. In [3], De studied the notion of $\phi$-symmetry with several examples for Kenmotsu manifolds. In 1977, Adati and Matsumoto defined Para-sasakian manifold and special Para-Sasakian manifolds [4], which are special classes of an almost para contact manifold introduced by sato [5].

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2 Preliminaries

A differentiable manifold $M$ of dimension $n$ is called a Lorentzian $\alpha$-Sasakian manifold if it admits a (1,1) tensor filed $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and Lorentzian metric $g$ which satisfy [4,7]

\begin{align*}
\phi^2 & = I + \eta \otimes \xi, \\
\eta(\xi) & = -1, \\
g(\phi X, \phi Y) & = g(X,Y) + \eta(X)\eta(Y), \\
\phi \xi & = 0, \quad \eta(\phi X) = 0, \\
g(X, \xi) & = \eta(X),
\end{align*}

for all $X, Y \in TM$. From the above relations it follows that a Lorentzian $\alpha$-Sasakian manifold satisfies

\begin{align*}
\nabla_X \xi & = -\alpha \phi X, \\
(\nabla_X \eta)Y & = -\alpha g(X,Y), \\
(\nabla_X \phi)Y & = \alpha g(X,Y)\xi - \alpha \eta(Y)X,
\end{align*}

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

Also, a Lorentzian $\alpha$-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any vector fields $X, Y$ where $a, b$ are functions on $M$.

Further, on such an From the above relations it follows that a Lorentzian $\alpha$-Sasakian manifold satisfies the following relations hold[7]

\begin{align*}
R(X, Y)\xi & = \alpha^2(\eta(Y)X + \eta(X)Y), \\
R(\xi, X)Y & = \alpha^2(g(X,Y)\xi + \eta(Y)X), \\
R(\xi, X)\xi & = \alpha^2(X + \eta(X)\xi), \\
S(X, \xi) & = (n-1)\alpha^2 \eta(X), \\
Q\xi & = (n-1)\alpha^2 \eta, \\
S(\xi, \xi) & = -(n-1)\alpha^2, \\
S(\phi X, \phi Y) & = S(X, Y) + (n-1)\alpha^2 \eta(X)\eta(Y),
\end{align*}

for any vector fields $X, Y, Z$, where $R(X, Y)Z$ is the curvature tensor, and $S$ is the Ricci tensor.

**Definition 2.1** An $n$-dimensional Lorentzian $\alpha$-Sasakian manifold is said to be an Einstein manifold if its Ricci tensor satisfies the condition

$$S(X, Y) = \lambda g(X, Y),$$

where $\lambda$ is a constant.

**Definition 2.2** A Lorentzian $\alpha$-Sasakian manifold is said to be Ricci-semi symmetric if its Ricci tensor satisfies the condition

\[ S(X, Y) = \lambda g(X, Y), \]
3 Main Results

In this section, we prove the following theorems:

**Theorem 3.1** Let $M$ be an $n$-dimensional Lorentzian $\alpha$-Sasakian manifold. If $M$ is Ricci semisymmetric then it is an $\eta$-Einstein manifold.

**Proof.** Suppose that $M$ is Ricci semisymmetric then in view of (2.18) we have

$$R(X,Y).S = 0,$$

for any vector fields $X$, $Y$.

**Definition 3.2** A Lorentzian $\alpha$-Sasakian manifold $M$ is said to be $\phi$-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = 0,$$

for all vector fields $X$ and $Y$ on $M$ and $S(X,Y) = g(QX,Y)$ [4].

If $X$ and $Y$ are orthogonal to $\xi$, then manifold is said to be locally $\phi$-Ricci symmetric.

**Theorem 3.3** An $n$-dimensional Lorentzian $\alpha$-Sasakian manifold is $\phi$-Ricci symmetric if and only if manifold is an Einstein manifold.

**Proof.** Suppose that the manifold is $\phi$-Ricci symmetric then in view of Definition 3.2 we have

$$\phi^2((\nabla_X Q)(Y)) = 0.$$

Using (2.1) in above equation we obtain

$$(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0.$$  

(3.4)

Taking inner product of (3.4) with $Z$ we get

$$g((\nabla_X Q)(Y),Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0,$$

which implies

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\[ g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta(\nabla_X Q(Y))\eta(Z) = 0, \]

which on simplifying gives
\[ g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta(\nabla_X Q(Y))\eta(Z) = 0. \] (3.5)

Replacing \( Y \) by \( \xi \) in (3.5) we get
\[ g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) + \eta(\nabla_X Q(\xi))\eta(Z) = 0. \] (3.6)

Using (2.4), (2.13) and (2.14) in (3.6) we obtain
\[-(n-1)\alpha^3 g(\phi X, Z) + \alpha S(\phi X, Z) + \eta(\nabla_X Q(\xi))\eta(Z) = 0. \] (3.7)

Replacing \( Z \) by \( \phi Z \) in (3.7) we get
\[ S(\phi X, \phi Z) = (n-1)\alpha^2 g(\phi X, \phi Z). \] (3.8)

Using (2.3) and (2.16) in (3.8) we obtain
\[ S(X, Z) = (n-1)\alpha^2 g(X, Z). \]

Therefore, the manifold is an Einstein manifold.

Next, suppose that the manifold is an Einstein manifold. Then in view of (2.17) we have
\[ S(X, Y) = \lambda g(X, Y), \] where \( S(X, Y) = g(QX, Y) \) and \( \lambda \) is constant. Hence \( QX = \lambda X \).
Therefore, we obtain \( \phi^2((\nabla_X Q(Y)) = 0. \) This completes the proof.

**Theorem 3.4** An \( n \)-dimensional \( (n > 3) \), Lorentzian \( \alpha \)-Sasakian manifold satisfying the condition \( S(X, \xi), R = 0 \) is an \( \eta \)-Einstein manifold.

**Proof.** Since \( S(X, \xi), R = 0 \) we have
\[ (S(X, \xi), R)(U, V)Z = 0, \]
which implies
\[ 0 = ((X \wedge_s \xi), R)(U, V)Z \]
\[ = (X \wedge_s \xi)R(U, V)Z + R((X \wedge_s \xi)U, V)Z \]
\[ + R(U, (X \wedge_s \xi)V)Z + R(U, V)(X \wedge_s \xi)Z, \] (3.9)

where endomorphism \( X \wedge_s Y \) is defined by
\[ (X \wedge_s Y)Z = S(Y, Z)X - S(X, Z)Y. \] (3.10)

Using (3.10) in (3.9) we get by virtue of (2.13)
\[ 0 = (n-1)\alpha^2[\eta(R(U, V)Z)X + \eta(U)\eta(R(X, V))Z \]
\[ + \eta(V)R(U, X)Z + \eta(Z)R(U, V)X] \]
\[ - S(X, R(U, V))Z - S(X, U)R(\xi, V)Z \]
\[ - S(X, V)R(U, \xi)Z - S(X, Z)R(U, V)\xi, \]

taking the inner product with \( \xi \) we obtain
\[ 0 = (n-1)\alpha^2[\eta(R(U, V)Z)\eta(X) + \eta(U)\eta(R(X, V))Z \]
\[ + \eta(V)\eta(R(U, X))Z + \eta(Z)\eta(R(U, V)X)] \]
\[ + S(X, R(U, V)Z) - S(X, U)\eta(R(\xi, V)Z \]
\[ - S(X, V)\eta(R(U, \xi))Z - S(X, Z)\eta(R(U, V)\xi). \]

Putting \( U = Z = \xi \) in the above equation an using (2.10)-(2.13) we get
\[ 0 = (n-1)\alpha^2[-2\alpha^2\eta(V)\eta(X) + \alpha^2 g(V, X) - \alpha^2 \eta(V)\eta(X)] \]
\[ + (n-1)\alpha^4 \eta(V)\eta(X) + \alpha^2 S(X,V), \]

with simplify of the last equation we have

\[ S(X,V) = -(n-1)\alpha^2 g(X,V) + 2(n-1)\alpha^2 \eta(X)\eta(V). \]

Therefore, in view of (2.9) manifold is an \( \eta \)-Einstein manifold. The proof is complete.

**Definition 3.5** A Lorentzian \( \alpha \)-Sasakian manifold \( M \) is said to be \( \phi \)-symmetric if

\[ \phi^5((\nabla_w R)(X,Y)Z) = 0, \]

for all vector fields \( X, Y, Z, W \) on \( M \) [6].

**Theorem 3.6** A \( \phi \)-symmetric Lorentzian \( \alpha \)-Sasakian manifold is an \( \eta \)-Einstein manifold.

**Proof.** If manifold is \( \phi \)-symmetric then in view of Definition 3.5 we have

\[ \phi^5((\nabla_w R)(X,Y)Z) = 0, \]

by virtue of (2.1) we get

\[ (\nabla_w R)(X,Y)Z + \eta((\nabla_w R)(X,Y)Z)\xi = 0, \]

taking inner product with \( U \), we obtain

\[ g((\nabla_w R)(X,Y)Z, U) + \eta((\nabla_w R)(X,Y)Z)g(\xi, U) = 0. \] (3.11)

Let \( \{ e_i \}, \ i = 1,2,...,n, \) be an orthonormal basis of tangent space at any point of the manifold. Then by putting \( X = U = e_i \) in (3.11) and taking summation over \( i, \ 1 \leq i \leq n, \) we have

\[ (\nabla_w S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_w R)(e_i,Y)Z)g(\xi, e_i) = 0. \]

Replacing \( Z = \xi \) in the above equation, we obtain

\[ (\nabla_w S)(Y,\xi) + \sum_{i=1}^{n} \eta((\nabla_w R)(e_i,Y)\xi)g(\xi, e_i) = 0. \] (3.12)

The second term of (3.12), takes the form

\[ \eta((\nabla_w R)(e_i,Y)\xi) = g(\nabla_w R(e_i,Y)\xi, \xi) - g(R(\nabla_w e_i,Y)\xi, \xi) \]

\[ - g(R(e_i,\nabla_w Y)\xi, \xi) - g(R(e_i,Y)\nabla_w \xi, \xi), \]

with simplify of the above equation we have

\[ \eta((\nabla_w R)(e_i,Y)\xi) = 0. \] (3.13)

The equations (3.12) and (3.13) imply that

\[ (\nabla_w S)(Y, \xi) = 0, \]

which gives

\[ \nabla_w (S(Y,\xi)) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi) = 0, \]

in view of (2.6) and (2.6) we obtain

\[ (n-1)\alpha^2 \nabla_w \eta(Y) - (n-1)\alpha^2 \eta(\nabla_w Y) + \alpha S(Y, \phi W) = 0. \] (3.14)

Replacing \( Y \) by \( \phi Y \) in (3.14) we get

\[ S(\phi Y, \phi W) = (n-1)\alpha g((\nabla_w \phi)Y, \xi). \] (3.15)

Using (2.2), (2.8) and (2.16) in the above equation we have
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\[ S(Y, W) = -(n-1)\alpha^2 g(W, Y) - 2(n-1)\alpha^2 \eta(Y)\eta(W). \]

This implies that manifold is an \( \eta \)-Einstein.

**Reference**


