On positive weak solutions for some nonlinear elliptic boundary value problems involving the p-Laplacian

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Abstract

This study concerns the existence of positive weak solutions to boundary value problems of the form

\[
\begin{cases}
-\Delta_p u = g(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}
\]

where $\Delta_p$ is the so-called p-Laplacian operator i.e. $\Delta_p z = div (|\nabla z|^{p-2} \nabla z)$, $p > 1$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with $\partial \Omega$ of class $C^2$, and connected, and $g(x, 0) < 0$ for some $x \in \Omega$ (semipositive problems). By using the method of sub-super solutions we prove the existence of the positive weak solution to special types of $g(x, u)$.

Keywords: Positive weak solutions, p-Laplacian, sub-super solution

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1 Introduction

In this paper we consider the existence of positive weak solution to boundary value problems of the form

\[
\begin{cases}
-\Delta_p u = g(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}
\]

where \(\Delta_p\) is the so-called p-Laplacian operator i.e. \(\Delta_p z = \text{div}(\lvert \nabla z \rvert^{p-2} \nabla z)\), \(p > 1\), \(\Omega\) is a smooth bounded domain in \(R^N (N \geq 2)\) with \(\partial \Omega\) of class \(C^2\), and connected, and \(g(x, 0) < 0\) for some \(x \in \Omega\) (semipositone problems). In particular, we first study the case when \(g(x, u) = a(x) u^{p-1} - b(x) u^{q-1} - ch(x)\), where \(q > p\) and \(a(x), b(x)\) are \(C^1(\bar{\Omega})\) functions that \(a(x)\) is allowed to be negative near the boundary of \(\Omega\), and \(b(x) > b_0 > 0\) for \(x \in \Omega\). Here \(h : \bar{\Omega} \rightarrow R\) is a \(C^1(\bar{\Omega})\) function satisfying \(h(x) \geq 0\) for \(x \in \Omega\), \(h(x) \neq 0\), and \(\max_{x \in \Omega} h(x) = 1\). We prove that there exists a \(c_0 = c_0(\Omega, a, b) > 0\) such that for \(0 < c < c_0\) there exists a positive solution.

Problems involving the “p-Laplacian” arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [10]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

The above equation arises in the studies of population biology of one species with \(u\) representing the concentration of the species or the population density, and \(ch(x)\) representing the rate of harvesting (see [7]).

In the earlier paper [1] we consider the problem (1) with \(p = 2\). The purpose of this paper is to extend this study to the p-Laplacian case. The case when \(p = 2\) (the Laplacian operator), \(a(x), b(x)\) are positive constants throughout \(\bar{\Omega}\), has been studied in [7]. Also recently in [8] the authors extend this study to the p-Laplacian case. In [3] the authors studied the case when \(c = 0\) (non-harvesting case), \(b(\bar{\Omega}) = 1\) for \(\bar{\Omega}\) and \(a(\bar{\Omega})\) is a positive function throughout \(\bar{\Omega}\). However the \(c > 0\) case is a semipositone problem \((g(x, 0) < 0)\) and studying positive solutions in this case is significantly harder. Here we consider the challenging semipositone case \(c > 0\). Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [2, 5]).

We next study the case when \(g(x, u) = \lambda m(x) f(u)\), where the weight \(m\) satisfying \(m \in C(\Omega)\) and \(m(x) \geq m_0 > 0\) for \(x \in \Omega\), \(f \in C^1[0, \rho]\) is a nondecreasing function for some \(\rho > 0\) such that \(f(0) < 0\) and there exist \(\alpha \in (0, \rho)\) such that \(f(t)(t - \alpha) \geq 0\) for \(t \in [0, \rho]\).
See [5] where positive solution is obtained for large \( \lambda \) when \( m(x) \equiv 1 \) for \( x \in \Omega \) and \( f \) is \( p \)-sublinear at infinity. We are interested in the existence of a positive solution in a range of \( \lambda \) without assuming any condition on \( f \) at infinity. Our approach is based on the method of sub-super solutions, see [3, 9].

2 Existence results

Let \( W_0^{1,s} = W_0^{1,s}(\Omega) \), \( s > 1 \), denote the usual Sobolev space. We give the definition of weak solution and sub-super solution of (1).

**Definition 2.1.** We say that \( u \in W_0^{1,p}(\Omega) \) is a weak solution to (1) if for any \( v \in W_0^{1,p} \) with \( v \geq 0 \) we have

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} g(x, u) \, v \, dx.
\]

However in this paper, we in fact study the existence of \( C^1(\overline{\Omega}) \) solutions that strictly positive in \( \Omega \).

**Definition 2.2.** We say that \( \psi \in W_0^{1,p}(\Omega) \) is a subsolution to (1) if

\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla v \, dx \leq \int_{\Omega} g(x, \psi) \, v \, dx,
\]

hold for all \( v \in W_0^{1,p} \) with \( v \geq 0 \).

**Definition 2.3.** We say that \( z \in W_0^{1,p}(\Omega) \) is a supersolution to (1) if

\[
\int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla v \, dx \geq \int_{\Omega} g(x, z) \, v \, dx,
\]

hold for all \( v \in W_0^{1,p} \) with \( v \geq 0 \).

Now if there exists sub and super solutions \( \psi \) and \( z \) respectively such that \( 0 \leq \psi \leq z \) for \( x \in \Omega \), then (1) has a positive solution \( u \in W_0^{1,p}(\Omega) \) such that \( \psi \leq u \leq z \) (see [3, 4]). We shall obtain the existence of positive weak solution to problem (1) by constructing a positive subsolution \( \psi \) and supersolution \( z \).

To precisely state our existence result we consider the eigenvalue problem

\[
\begin{cases}
-\Delta_p \phi = \lambda |\phi|^{p-2} \phi, & x \in \Omega, \\
\phi = 0, & x \in \partial \Omega.
\end{cases}
\]

(2)

Let \( \phi_1 \in C^1(\overline{\Omega}) \) be the eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of (2) such that \( \phi_1(x) > 0 \) in \( \Omega \), and \( ||\phi_1||_\infty = 1 \). It can be shown that \( \frac{\partial \phi_1}{\partial n} < 0 \) on \( \partial \Omega \) and hence, depending on \( \Omega \), there exist positive constants \( k, \eta, \mu \) such that

\[
\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -k, \quad x \in \Omega_\eta,
\]

(3)
\[ \phi_1 \geq \mu, \quad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\eta, \]  
with \( \bar{\Omega}_\eta = \{ x \in \Omega \mid d(x, \partial \Omega) \leq \eta \} \). Further assume that there exists a constants \( a_0, a_1 > 0 \) such that \( a(x) \geq -a_0 \) in \( \bar{\Omega}_\eta \) and \( a(x) \geq a_1 \) in \( \Omega_0 = \Omega \setminus \bar{\Omega}_\eta \).

We will also consider the unique solution, \( \zeta \in C^1(\bar{\Omega}) \), of the boundary value problem

\[
\begin{cases}
-\Delta_p \zeta = 1, & x \in \Omega, \\
\zeta = 0, & x \in \partial \Omega,
\end{cases}
\]

to discuss our existence result. It is known that \( \zeta > 0 \) in \( \Omega \) and \( \frac{\partial \zeta}{\partial n} < 0 \) on \( \partial \Omega \).

First we obtain the existence of positive weak solution of (1) in the case when \( g(x,u) = a(x) u^{p-1} - b(x) u^{q-1} - ch(x) \).

**Theorem 2.4.** Suppose that \( a_0 < k (p/(p-1))^{p-1} \) and \( \lambda_1 (p/(p-1))^{p-1} < a_1 \). Then there exists \( c_0 = c_0(\Omega, a_0, a_1, b) > 0 \) such that if \( 0 < c < c_0 \) then the problem (1) has a positive solution \( u \).

**Proof.** To obtain the existence of positive weak solution to problem (1), we constructing a positive subsolution \( \psi \) and supersolution \( z \). We shall verify that \( \psi = \delta \phi_1^{1/(p-1)} \) is a subsolution of (1), where \( \delta > 0 \) is small and specified later (note that \( ||\psi||_{\infty} \leq \delta \)). Let the test function \( w \in W_0^{1,p} \) with \( w \leq 0 \). A calculation shows that

\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w = \delta^{p-1} (p/p-1)^{p-1} \int_{\Omega} \phi_1 |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla w \ dx \\
= \delta^{p-1} (p/p-1)^{p-1} \left\{ \int_{\Omega} |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla (\phi_1 w) \ dx - \int_{\Omega} |\nabla \phi_1|^p w \ dx \right\} \\
= \delta^{p-1} (p/p-1)^{p-1} \int_{\Omega} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \ dx.
\]

Thus \( \psi \) is a subsolution if

\[
\delta^{p-1} (p/p-1)^{p-1} \int_{\Omega} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \ dx \leq \int_{\Omega} (a(x) \psi^{p-1} - b(x) \psi^{q-1} - ch(x)) w \ dx.
\]

Now \( \lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -k \) in \( \bar{\Omega}_\eta \), and therefore

\[
\delta^{p-1} (p/p-1)^{p-1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq -k \delta^{p-1} (p/p-1)^{p-1} \\
\leq -a_0 \delta^{p-1} - ||b||_{\infty} \delta^{q-1} - c,
\]

if

\[
\delta < \theta_1 = (\frac{k (p/p-1)^{p-1} - a_0}{||b||_{\infty}})^{1/q-p},
\]

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\[ c \leq \hat{c}(\delta) = \delta^{p-1}(k(p/p - 1)^{p-1} - a_0 - ||b||_\infty \delta^{q-p}). \]

Clearly \( \hat{c}(\delta) > 0 \).

Furthermore, we note that \( \phi_1 \geq \mu > 0 \) in \( \Omega_0 = \Omega \setminus \Omega_{\eta} \), and therefore

\[ \delta^{p-1}(p/p - 1)^{p-1}(\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq \lambda_1 \delta^{p-1}(p/p - 1)^{p-1} \]
\[ \leq a_1 \delta^{p-1} \phi_1 - ||b||_\infty \delta^{q-1} - c, \]

if

\[ \delta < \theta_2 = \left( \frac{(a_1 - (p/p - 1)^{p-1} \lambda_1) \mu^p}{||b||_\infty} \right)^{1/q-p}, \]

\[ c \leq \hat{c}(\delta) = \delta^{p-1}(a_0 - (p/p - 1)^{p-1} \lambda_1) \mu^p - ||b||_\infty \delta^{q-p}). \]

Clearly \( \hat{c}(\delta) > 0 \). Choose \( \theta = \min\{\theta_1, \theta_2\} \) and \( \delta = \theta/2 \). Then simplifying, both \( \hat{c} \) and \( \hat{c} \) are greater than \( (\frac{\theta}{2})^{q-1}(2^{q-p} ||b||_\infty - ||b||_\infty) \). Hence if \( c \leq (\frac{\theta}{2})^{q-1}(2^{q-p} ||b||_\infty - ||b||_\infty) = c_0(\Omega, a_0, a_1, b) \) then \( \psi \) is a subsolution.

Next, we construct a supersolution \( z \) of (1). We denote \( z = N \zeta(x) \), where the constant \( N > 0 \) is large and to be chosen later. We shall verify that \( z \) is a supersolution of (1). To this end, let \( w(x) \in W_0^{1,p}(\Omega) \) with \( w \geq 0 \). Then we have

\[ \int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx = N^{p-1} \int_\Omega |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla w \, dx \]
\[ = N^{p-1} \int_\Omega w \, dx. \]

Thus \( z \) is a supersolution if

\[ N^{p-1} \int_\Omega w \, dx \geq \int_\Omega (a(x) z^{p-1} - b(x) z^{q-1} - ch(x)) w \, dx, \]

and therefore if \( N \geq N_0^{1/(p-1)} \) where \( N_0 = \sup_{[0, ||a||_\infty/b_0]^{1/q-p}}(||a||_\infty v^{p-1} - b_0 v^{q-1}) \), we have

\[ \int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx \geq \int_\Omega (a(x) z^{p-1} - b(x) z^{q-1} - ch(x)) w \, dx, \]

and hence \( z \) is supersolution of (1). Since \( \zeta > 0 \) and \( \partial \zeta/\partial n < 0 \) on \( \partial \Omega \), we can choose \( N \) large enough so that \( \psi \leq z \) is also satisfied. Hence Theorem 2.4 is proven. \( \square \)

Now, we obtain the existence of positive weak solution of (1) in the case when \( g(x,u) = \lambda m(x) f(u) \). Assume that there exist positive constants \( r_1, r_2 \in (\alpha, \rho] \) satisfying:

\[ (H.1) \frac{r_2}{r_1} \geq \max\{\frac{\lambda_1^{1/p-1}(p||\zeta||_\infty m^{p/(p-1)})}{(\frac{m_0}{m_0})^{p/(p-1)}}(p||\zeta||_\infty f(r_1))^{1/p-1}\}, \]

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\((H.2)\) \(k f(r_1) > \lambda_1 |f(0)|.\)

**Theorem 2.5.** Let \((H.1), (H.2)\) hold. Then there exist \(\lambda < \tilde{\lambda}\) such that \((1)\) has a positive solution for \(\lambda \in [\lambda_*, \tilde{\lambda}]\).

**Proof.** Let \(\lambda_1, \phi_1\), be as before. We now construct our positive subsolution. Let \(\psi = r_1 \mu^{p/(p-1)} \phi_1^{p/(p-1)}\). Let the test function \(w(x) \in W^{1,p}_0(\Omega)\) with \(w \geq 0\). Using a calculation similar to the one in the proof of Theorem 2.4, we have

\[
\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w = \left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \int_{\Omega} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \, dx.
\]

Thus \(\psi\) is a subsolution if

\[
\left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \int_{\Omega} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \, dx \leq \lambda \int_{\Omega} m(x) f(\psi) \, w \, dx,
\]

Now \(\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -k\) in \(\Omega_\eta\), and therefore

\[
\left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \left(\lambda_1 \phi_1^p - |\nabla \phi_1|^p\right) \leq -k \left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \leq \lambda m(x) f(\psi),
\]

if

\[
\lambda \leq \tilde{\lambda} = \frac{k \left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)}}{m_0 |f(0)|}.
\]

Furthermore, we note that \(\phi_1 \geq \mu > 0\) in \(\Omega_0 = \Omega \setminus \bar{\Omega}_\eta\), and therefore

\[
\psi = r_1 \mu^{p/(p-1)} \phi_1^{p/(p-1)} \geq r_1 \mu^{p/(p-1)} \mu^{p/(p-1)} = r_1,
\]

thus \(f(\psi) \geq f(r_1)\). Hence if

\[
\lambda \geq \lambda_* = \frac{\lambda_1 \left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)}}{m_0 f(r_1)},
\]

we have

\[
\left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \left(\lambda_1 \phi_1^p - |\nabla \phi_1|^p\right) \leq \lambda_1 \left(\frac{p}{p-1}\right) r_1 \mu^{p/(p-1)} \leq \lambda m_0 f(r_1) \leq \lambda m(x) f(\psi).
\]
We get \( \lambda_* < \hat{\lambda} \) by using \((H.2)\). Therefore if \( \lambda_* \leq \lambda \leq \hat{\lambda} \), then \( \psi \) is subsolution.

Next, we construct a supersolution \( z \) of \((1)\) such that \( z \geq \psi \). We denote \( z = \frac{r_2}{||\zeta||_\infty} \zeta(x) \). We shall verify that \( z \) is a super solution of \((1)\). To this end, let \( w(x) \in W^{1,p}_0(\Omega) \) with \( w \geq 0 \). Then we have

\[
\int_\Omega |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx = \left( \frac{r_2}{||\zeta||_\infty} \right)^{p-1} \int_\Omega w \, dx.
\]

(6)

Thus \( z \) is a super solution if

\[
\left( \frac{r_2}{||\zeta||_\infty} \right)^{p-1} \int_\Omega w \, dx \geq \lambda \int_\Omega m(x) f(z) w \, dx.
\]

But \( f(z) \leq f(r_2) \) and hence \( z \) is a super solution if

\[
\lambda \leq \hat{\lambda} = \frac{(r_2/||\zeta||_\infty)^{p-1}}{||m||_\infty |f(r_2)|}.
\]

We easily see that \( \lambda_* < \hat{\lambda} \), by using \((H.1)\). Finally, using \((5),(6)\) and the weak comparoson principle \([4]\), we see that \( \psi \leq z \) in \( \Omega \) when \((H.1)\) is satisfied. Therefore \((1)\) has a positive solution for \( \lambda \in [\lambda_*, \hat{\lambda}] \), where \( \hat{\lambda} = \min\{\lambda, \hat{\lambda}\} \). This completes the proof of Theorem 2.5. \( \square \)

**Remark 2.6.** Theorem 2.5 holds no matter what the growth condition of \( f \) is, for large \( u \). Namely, \( f \) could satisfy p-superlinear or p-linear growth condition at infinity.

**References**


