Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci. 16 (2016), 211–217
Research Article

Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Khan type fixed point theorems in a generalized metric space

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Abstract

Existence and uniqueness of fixed points are established for a mapping satisfying a new type of contractive condition involving a rational expression on a generalized metric space. Some main results by Ahmad et al. [J. Ahmad, M. Arshad, C. Vetro, Int. J. Anal., **2013** (2013), 6 pages] are extended and generalized, also several particular cases and an illustrative example are given.

Keywords: Fixed point, metric space.
2010 MSC: 74H10, 54H25.

1. Introduction and preliminaries

In the mid-sixties ten, fixed points results dealing with general contractive conditions with rational expressions were appeared. One of the well-known works in this direction were established by Khan [4]. Fisher [3] gave a revised version of Khan as follows:

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Received 2016-02-14

Theorem 1.1 ([3]). *Let (X, d) be a complete metric space and let $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq \begin{cases} k\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)}, & if\ d(x,Ty)+d(Tx,y) \neq 0, \\ 0, & if\ d(x,Ty)+d(Tx,y) = 0, \end{cases} \tag{1.1}$$

where $k \in [0, 1)$ and $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Ahmad et al. [1] gave a new version of Theorem 1.1 in the setting of generalized metric spaces as follows:

Theorem 1.2. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a self-mapping such that for all $x, y \in X$*

$$d(Tx,Ty) \leq \begin{cases} \delta d(x,y) + \zeta\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)}, & if d(x,Ty)+d(Tx,y) \neq 0, \\ 0, & if d(x,Ty)+d(Tx,y) = 0, \end{cases} \tag{1.2}$$

for all $x, y \in X$ and $x \neq y$, and for some $\delta, \zeta \in [0, 1)$ with $\delta + \zeta < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

The aim of this paper is to give a new version of Theorem 1.1 in the setting of generalized metric spaces. The following definitions will be needed in the sequel.

Definition 1.3 ([2]). Let X be a nonempty set and d: $X \times X \to \mathbb{R}^+$ a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, which are different from x and y, one has

$(GM1)$ $d(x, y) = 0$ if and only if $x = y$;

$(GM2)$ $d(x, y) = d(y, x)$;

$(GM3)$ $d(x, z) \leq d(x, u) + d(u, v) + d(v, z)]$.

Then d is called generalized metric and the pair (X, d) is called generalized metric space (or shortly GMS).

For some examples about generalized metric space, we refer readers to [1, 5, 6].

Definition 1.4 ([2]). Let (X, d) be a GMS, $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$. Then

(i) We say that $\{x_n\}_{n=1}^{\infty}$ is GMS convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

(ii) We say that $\{x_n\}_{n=1}^{\infty}$ is a GMS Cauchy sequence if and only if, for each $\epsilon > 0$, there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > n(\epsilon)$.

(iii) (X, d) is called GMS complete if every GMS Cauchy sequence is GMS convergent in X.

2. Main results

Theorem 2.1. *Let (X, d) be a complete GMS and let $T : X \to X$ be a self-mapping such that*

$$d(Tx,Ty) \leq \begin{cases} \gamma \max\left\{d(x,y), \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}\right\}, & if \max\{d(x,Ty), d(Tx,y)\} \neq 0, \\ 0, & if \max\{d(x,Ty), d(Tx,y)\} = 0, \end{cases} \tag{2.1}$$

for all $x, y \in X$ and $x \neq y$, and for some $\gamma \in [0, 1)$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Proof. Let $x_0 = x \in X$. Put $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If, there exists $n \in \mathbb{N}$ such that $x_n = x_{n-1}$, then x_{n-1} is a fixed point of T. This completes the proof. Therefore, we suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. We shall divide the proof into two cases.

Cases 1. Assume that

$$\max\{d(x_m, Tx_n), d(Tx_m, x_n)\} \neq 0$$

for all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then from (2.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \gamma \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n) + d(x_n, Tx_n)d(x_n, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\}} \right\} \\ &= \gamma \max\left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}} \right\} \\ &= \gamma d(x_{n-1}, x_n). \end{aligned} \tag{2.2}$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is monotonic nonincreasing and bounded below. So, there exists $\zeta \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \zeta = \lim_{n\to\infty} d(x_{n-1}, x_n). \tag{2.3}$$

We claim that $\zeta = 0$. Suppose by the contrary that $\zeta > 0$. Taking limits as $n \to \infty$ to each side of the (2.2), we get $\zeta \leq \gamma\zeta < \zeta$. It is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

A gain using inequality (2.1), we have

$$\begin{aligned} &d(x_n, x_{n+2}) \\ &\quad= d(Tx_{n-1}, Tx_{n+1}) \\ &\quad\leq \gamma \max\left\{ d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1})d(x_{n+1}, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_{n+1}), d(Tx_{n-1}, x_{n+1})\}} \right\} \\ &\quad= \gamma \max\left\{ d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+2}) + d(x_{n+1}, x_{n+2})d(x_{n+1}, x_n)}{\max\{d(x_{n-1}, x_{n+2}), d(x_n, x_{n+1})\}} \right\} \\ &\quad\leq \gamma \max\left\{ d(x_{n-1}, x_{n+1}), \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+2})}{d(x_{n-1}, x_{n+2})} + \frac{d(x_{n+1}, x_{n+2})d(x_{n+1}, x_n)}{d(x_n, x_{n+1})} \right\} \\ &\quad= \gamma \max\left\{ d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2}) \right\} \\ &\quad\leq \gamma \max\left\{ d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1}), d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2}) \right\} \\ &\quad= \gamma \left[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})) \right]. \end{aligned}$$

It follows that

$$d(x_n, x_{n+2}) \leq \frac{1}{1-\gamma} \left[d(x_{n-1}, x_n) + d(x_{n+2}, x_{n+1}) \right].$$

So from (2.4), we get

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.5}$$

Now, we claim that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Observe that

$$\begin{aligned}\epsilon \leq d(x_{p(n)}, x_{q(n)}) &\leq d(x_{p(n)}, x_{p(n)-2}) + d(x_{p(n)-2}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})\\ &= d(x_{p(n)}, x_{p(n)-2}) + d(x_{p(n)-2}, x_{p(n)-1}) + \epsilon.\end{aligned}$$

It follows from (2.4) and (2.5) that

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{2.7}$$

From (2.1), we have

$$\begin{aligned}\epsilon &\leq d(x_{p(n)+1}, x_{q(n)+1})\\ &= d(Tx_{p(n)}, Tx_{q(n)}))\\ &\leq \gamma \max\left\{d(x_{p(n)}, x_{q(n)}), \frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}}\right\}\\ &= \gamma \max\left\{d(x_{p(n)}, x_{q(n)}), \frac{d(x_{p(n)}, x_{p(n)+1})d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)}, x_{q(n)+1})d(x_{q(n)}, x_{p(n)+1})}{\max\{d(x_{p(n)}, x_{q(n)+1}), d(x_{p(n)+1}, x_{q(n)})\}}\right\}\\ &\leq \gamma \max\left\{d(x_{p(n)}, x_{q(n)}), \frac{d(x_{p(n)}, x_{p(n)+1})d(x_{p(n)}, x_{q(n)+1})}{d(x_{p(n)}, x_{q(n)+1})} + \frac{d(x_{q(n)}, x_{q(n)+1})d(x_{q(n)}, x_{p(n)+1})}{d(x_{p(n)+1}, x_{q(n)})}\right\}\\ &= \gamma \max\left\{d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, x_{p(n)+1}) + d(x_{q(n)}, x_{q(n)+1})\right\}.\end{aligned}$$

It follows from (2.4) and (2.7) that $\epsilon \leq \gamma\epsilon < \epsilon$. This contradiction shows that $\{x_n\}$ is a cauchy sequence. By Completeness of (X, d), $\{x_n\}$ converges to some point x^* in X. Therefore

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \tag{2.8}$$

Observe that

$$\begin{aligned}d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, Tx^*)\\ &\leq d(x^*, x_{n+1}) + d(x_{n+1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x^*) + d(x^*, Tx^*)\\ &\leq 2d(x^*, x_{n+1}) + 2d(x_{n+1}, x_n) + d(x^*, Tx^*).\end{aligned}$$

It follows from (2.4) and (2.8)that

$$\lim_{n\to\infty} d(x_n, Tx^*) = d(x^*, Tx^*). \tag{2.9}$$

On the other hand, from (2.1), we get

$$\begin{aligned}d(x_{n+1}, Tx^*) &= d(Tx_n, Tx^*)\\ &\leq \gamma \max\left\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\right\}\\ &\leq \gamma \max\left\{d(x_n, x^*), \frac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, x_{n+1})}{\max\{d(x_n, Tx^*), d(x_{n+1}, x^*)\}}\right\}.\end{aligned} \tag{2.10}$$

So from (2.4), (2.8), (2.9) and taking limits as $n \to \infty$ to each side of (2.10), we have $d(x^*, Tx^*) = 0$. Now, we show that T has a unique fixed point. For this, we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore

$$\max\{d(x^*, Ty^*), d(Tx^*, y^*)\} = d(x^*, y^*) > 0.$$

So from (2.1), we get

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq \gamma \max\left\{d(x^*, y^*), \frac{d(x^*, Tx^*)d(x^*, Ty^*) + d(y^*, Ty^*)d(y^*, Tx^*)}{\max\{d(x^*, Ty^*), d(Tx^*, y^*)\}}\right\} \\ &\leq \gamma \max\left\{d(x^*, y^*), \frac{d(x^*, x^*)d(x^*, y^*) + d(y^*, y^*)d(y^*, x^*)}{\max\{d(x^*, y^*), d(x^*, y^*)\}}\right\} \\ &= \gamma d(x^*, y^*). \end{aligned}$$

Since $\gamma \in [0, 1)$. This leads to a contradiction and hence $x^* = y^*$. This complete the proof.

Cases 2. Assume that there exists $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that

$$\max\{d(x_m, Tx_n), d(Tx_m, x_n)\} = 0.$$

By condition (2.1), it follows that $d(Tx_m, Tx_n) = 0$ and hence $x_n = Tx_m = Tx_n = x_m$. This completes the proof of the existence of a fixed point of T. The uniqueness follows as in Case 1. □

Theorem 2.2. *Let (X, d) be a complete GMS and let $T : X \to X$ be a self-mapping such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \begin{cases} \delta d(x, y) + \zeta \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}, & if \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ 0, & if \max\{d(x, Ty), d(Tx, y)\} = 0, \end{cases}$$

for all $x, y \in X$ and $x \neq y$, and for some $\delta, \zeta \in [0, 1)$ such that $\delta + \zeta < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Proof. Since

$$\begin{aligned} &\delta d(x, y) + \zeta \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}} \\ &\qquad \leq (\delta + \zeta) \max\left\{d(x, y), \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}}\right\}. \end{aligned}$$

So by taking $\gamma = \delta + \zeta$ in Theorem 2.1, the proof is complete. □

Theorem 2.3 ([1]). *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a self-mapping such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \begin{cases} \delta d(x, y) + \zeta \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)}, & if\, d(x, Ty) + d(Tx, y) \neq 0, \\ 0, & if\, d(x, Ty) + d(Tx, y) = 0, \end{cases} \tag{2.11}$$

for all $x, y \in X$ and $x \neq y$, and for some $\delta, \zeta \in [0, 1)$ with $\delta + \zeta < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*.

Proof. Since

$$\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)} \leq \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}.$$

So from Theorem 2.2, the proof is complete. □

Example 2.4. Let $X=\{0,1,2,3\}$ and define $d: X\times X \to \mathbb{R}$ as follows:

$$\begin{aligned}
&d(0,0)=d(1,1)=d(2,2)=d(3,3)=0,\\
&d(0,3)=d(3,0)=d(2,3)=d(3,2)=d(1,2)=d(2,1)=1,\\
&d(0,2)=d(2,0)=d(1,3)=d(3,1)=2.1,\\
&d(0,1)=d(1,0)=1.1.
\end{aligned}$$

Then, (X,d) is a complete GMS. Let $T\colon X\to X$ be defined by

$$Tx=\begin{cases} 0, & if\ x\neq 3,\\ 1, & if\ x=3.\end{cases} \tag{2.12}$$

Now we consider the following cases:
Case1. Let $(x,y)\in\{(0,1),(1,0)\}$, then

$$\begin{aligned}
&d(T0,T1)=0, d(0,1)=1.1, d(0,T1)+d(T0,1)=1.1,\\
&d(0,T0)d(0,T1)+d(1,T1)d(1,T0)=1.21, \max\{d(0,T1),d(T0,1)\}=1.1.
\end{aligned}$$

Case2. Let $(x,y)\in\{(0,2),(2,0)\}$, then

$$\begin{aligned}
&d(T0,T2)=0, d(0,2)=2.1, d(0,T2)+d(T0,2)=2.1,\\
&d(0,T0)d(0,T2)+d(2,T2)d(2,T0)=4.4, \max\{d(0,T2),d(T0,2)\}=2.1.
\end{aligned}$$

Case3. Let $(x,y)\in\{(0,3),(3,0)\}$, then

$$\begin{aligned}
&d(T0,T3)=1.1, d(0,3)=1, d(0,T3)+d(T0,3)=2.1,\\
&d(0,T0)d(0,T3)+d(3,T3)d(3,T0)=2.1, \max\{d(0,T3),d(T0,3)\}=1.1.
\end{aligned}$$

Case4. Let $(x,y)\in\{(1,2),(2,1)\}$, then

$$\begin{aligned}
&d(T1,T2)=0, d(1,2)=1, d(1,T2)+d(T1,2)=3.1,\\
&d(1,T1)d(1,T2)+d(2,T2)d(2,T1)=5.62, \max\{d(1,T2),d(T1,2)\}=2.1.
\end{aligned}$$

Case5. Let $(x,y)\in\{(1,3),(3,1)\}$, then

$$\begin{aligned}
&d(T1,T3)=1.1, d(1,3)=2.1, d(1,T3)+d(T1,3)=1,\\
&d(1,T1)d(1,T3)+d(3,T3)d(3,T1)=2.1, \max\{d(1,T3),d(T1,3)\}=1.
\end{aligned}$$

Case6. Let $(x,y)\in\{(2,3),(3,2)\}$, then

$$\begin{aligned}
&d(T2,T3)=1.1, d(2,3)=1, d(2,T3)+d(T2,3)=2,\\
&d(2,T2)d(2,T3)+d(3,T3)d(3,T2)=3.1, \max\{d(2,T3),d(T2,3)\}=1.
\end{aligned}$$

In **Case3**, for all $\delta, \zeta \in [0,1)$ such that $\delta + \zeta < 1$, we have

$$\delta d(0,3) + \zeta \frac{d(0,T0)d(0,T3) + d(3,T3)d(3,T0)}{d(0,T3) + d(3,T0)} = \delta + \zeta < 1 < 1.1 = d(T0,T3).$$

This proves that T does not satisfy in assumption of Theorem 11 of [1]. However in all cases, for $\lambda = \frac{69}{80}$, $\zeta = \frac{1}{8}$ and $\delta \in [0.57, 1)$, we have

$$d(Tx,Ty) \leq \lambda d(x,y) + \zeta \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(y,Tx)\}},$$

and

$$d(Tx,Ty) \leq \delta \max\{d(x,y), \frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty), d(y,Tx)\}}\}.$$

So by Theorem 2.1 or Theorem 2.2, 0 is the unique fixed point of T.

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