A fixed point theorem in $S_b$-metric spaces

Nizar Souayah*, Nabil Mlaiki

*Department of Natural Sciences, Community College of Riyadh, King Saud University, Riyadh, Saudi Arabia.
Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia.

Abstract

In this paper, we introduce an interesting extension of the $S$-metric spaces called $S_b$-metric spaces, in which we show the existence of fixed point for a self mapping defined on such spaces. We also prove some results on the topology of the $S_b$-metric spaces. ©2016 All rights reserved.

Keywords: Functional analysis, $S_b$-metric space, common fixed point.

2010 MSC: 54H25, 47H10.

1. Introduction

The concept of metric spaces has been generalized in many ways. Bakhtin [2] introduced the $b$-metric space, in which many researchers treated the fixed point theory. Czerwick [5] extended the Banach principle contraction and its generalizations under different contractions [1, 4, 6, 7, 10, 15, 16, 17, 18] and [19].

Several authors have investigated the $S$-metric space and generalized many results related to the existence of fixed point, see [8, 9, 11, 12, 14] and [20]. However, no work has extended the fixed point problem from the $b$-metric spaces to the $S$-metric spaces.

Inspired by the work of Bakhtin in [2], we first introduce the $S_b$-metric space as a generalization of the $b$-metric space, and then we prove some fixed point results under different types of contractions in a complete $S_b$-metric space.

Recall the definitions of the $b$-metric space and the $S$-metric space.

**Definition 1.1** ([2]). Let $X$ be a nonempty set. A $b$-metric on $X$ is a function $d : X^2 \rightarrow [0, \infty)$ if there exists a real number $s \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

\[ d(x, y) \leq sd(x, z) + sd(y, z) - d(z, z) \]

*Corresponding author
Email addresses: nizar.souayah@yahoo.fr (Nizar Souayah), nmlaiki2012@gmail.com (Nabil Mlaiki)

Received 2016-01-19
(i) \(d(x, y) = 0\) if and only if \(x = y\),

(ii) \(d(x, y) = d(y, x)\),

(iii) \(d(x, z) \leq s[d(x, y) + d(y, z)]\).

The pair \((X, d)\) is called a \(b\)-metric space.

**Definition 1.2** (I3). Let \(X\) be a nonempty set. An \(S\)-metric on \(X\) is a function \(S : X^3 \to [0, \infty)\) that satisfies the following conditions, for all \(x, y, z, t \in X\):

(i) \(S(x, y, z) = 0\) if and only if \(x = y = z\),

(ii) \(S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)\).

The pair \((X, S)\) is called an \(S\)-metric space.

Now, we give the definition of the \(S_b\)-metric space.

**Definition 1.3.** Let \(X\) be a nonempty set and let \(s \geq 1\) be a given real number. A function \(S_b : X^3 \to [0, \infty)\) is said to be \(S_b\)-metric if and only if for all \(x, y, z, t \in X\) the following conditions hold:

(i) \(S_b(x, y, z) = 0\) if and only if \(x = y = z\),

(ii) \(S_b(x, y, z) = S_b(y, y, x)\) for all \(x, y \in X\),

(iii) \(S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]\).

The pair \((X, S_b)\) is called a \(S_b\)-metric space.

**Remark 1.4.** Note that the class of \(S_b\)-metric spaces is larger than the class of \(S\)-metric spaces. Indeed, every \(S\)-metric space is an \(S_b\)-metric space with \(s = 1\). However, the converse is not always true.

**Example 1.5.** Let \(X\) be a nonempty set and \(\text{card}(X) \geq 5\). Suppose \(X = X_1 \cup X_2\) a partition of \(X\) such that \(\text{card}(X_1) \geq 4\). Let \(s \geq 1\). Then

\[
S_b(x, y, z) = \begin{cases} 
0 & \text{if } x = y = z = 0 \\
3s & \text{if } (x, y, z) \in X_1^3 \\
1 & \text{if } (x, y, z) \notin X_1^3 
\end{cases}
\]

for all \(x, y, z \in X\) \(S_b\) is a \(S_b\)-metric on \(X\) with coefficient \(s \geq 1\).

**Proof.**

i) If \(x = y = z\) then \(S_b(x, y, z) = 0\). Thus the first assertion of the definition of the \(S_b\)-metric space is satisfied.

ii) Let’s prove the triangle inequality: \(S_b(x, y, z) \leq s[S_b(x, x, t) + S_b(y, y, t) + S_b(z, z, t)]\) (*).

- **Case 1:** If \((x, y, z) \notin X_1^3\). We have \(S_b(x, y, z) = 1\) \(S_b(x, x, t) \geq 1\), \(S_b(y, y, t) \geq 1\), and \(S_b(z, z, t) \geq 1\), for all \(t \in X\). Thus (*) is holds \((1 \leq 3s)\).
\textbf{Case 2:} If \((x, y, z) \in X^3_1\). We distinguish two sub-cases:

- if \(t \in X_1\), \((*)\) is satisfied since \(S_b(x, y, z) = S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 3s\).
- if \(t \notin X_1\), we have \(S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 1\) and \(S_b(x, y, z) = 3s\). Then, \((*)\) holds.

\begin{definition}
Let \((X, S_b)\) be an \(S_b\)-metric space and \(\{x_n\}\) be a sequence in \(X\). Then

(i) A sequence \(\{x_n\}\) is called convergent if and only if there exists \(z \in X\) such that \(S_b(x_n, x_n, z) \rightarrow 0\) as \(n \rightarrow \infty\). In this case we write \(\lim_{n \rightarrow \infty} x_n = z\).

(ii) A sequence \(\{x_n\}\) is called a Cauchy sequence if and only if \(S_b(x_n, x_n, x_m) \rightarrow 0\) as \(n, m \rightarrow \infty\).

(iii) \((X, S_b)\) is said to be a complete \(S_b\)-metric space if every Cauchy sequence \(\{x_n\}\) converges to a point \(x \in X\) such that
\[
\lim_{n,m \rightarrow \infty} S_b(x_n, x_n, x_m) = \lim_{n,m \rightarrow \infty} S_b(x_n, x_n, x) = S_b(x, x, x).
\]

(iv) Define the diameter of a subset \(Y\) of \(X\) by
\[
diam(Y) := \sup\{S_b(x, y, z) \mid x, y, z \in Y\}.
\]
\end{definition}

\begin{definition}[\textbf{3}]\textbf{.}
(i) Let \(E\) be a nonempty set and \(T : E \rightarrow E\) a selfmap. We say that \(x \in E\) is a fixed point of \(T\) if \(T(x) = x\).

(ii) Let \(E\) be any set and \(T : E \rightarrow E\) a selfmap. For any given \(x \in E\), we define \(T^n(x)\) inductively by \(T^0(x) = x\) and \(T^{n+1}(x) = T(T^n(x))\); we recall \(T^n(x)\) the \(n\)th iterative of \(x\) under \(T\). For any \(x_0 \in X\), the sequence \(\{x_n\}_{n \geq 0} \subset X\) given by
\[
x_n = T^n x_0, \ n = 1, 2, ... \tag{1.1}
\]
is called the sequence of successive approximations with the initial value \(x_0\). It is also known as the Picard iteration starting at \(x_0\).

\end{definition}

2. Main result

\begin{theorem}
Let \((X, S_b)\) be a complete \(S_b\)-metric space and \(T\) be a continuous self mapping on \(X\) satisfy
\[
S_b(Tx, Ty, Tz) \leq \psi[S_b(x, y, z)] \text{ for all } x, y, z \in X, \tag{2.1}
\]
where \(\psi : [0, +\infty) \rightarrow [0, +\infty)\) is an increasing function such that
\[
\lim_{n \rightarrow \infty} \psi^n(t) = 0 \text{ for each fixed } t > 0.
\]
Then \(T\) has a unique fixed point in \(X\).
\end{theorem}
Proof. Let \( x \in X \) and \( \epsilon > 0 \). Let \( n \) be a natural number such that \( \psi^n(\epsilon) < \frac{\epsilon}{2s} \). Let \( F = T^n \) and \( x_k = F^k(x) \) for \( k \in \mathbb{N} \). Then for \( x, y \in X \) and \( \alpha = \psi^n \) we have

\[
S_b(Fx, Fx, FY) \leq \psi^n(S_b(x, x, y)) = \alpha(S_b(x, x, y)).
\]

Hence, for \( k \in \mathbb{N} \)
\[
S_b(x_{k+1}, x_{k+1}, x_k) \rightarrow 0 \text{ as } k \rightarrow \infty.
\]

Therefore, let \( k \) be such that
\[
S_b(x_{k+1}, x_{k+1}, x_k) < \frac{\epsilon}{2s}.
\]

Let’s define the ball \( B(x_k, \epsilon) \) such that for every \( z \in B(x_k, \epsilon) := \{y \in X | S_b(x_k, x_k, y) \leq \epsilon\} \). Note that \( x_k \in B(x_k, \epsilon) \), therefore \( B(x_k, \epsilon) \neq \emptyset \). Hence, for all \( z \in B(x_k, \epsilon) \) we have

\[
S_b(Fz, Fz, Fx_k) \leq \alpha(S_b(x_k, x_k, z))
\]

\[
\leq \alpha(\epsilon) = \psi^n(\epsilon) < \frac{\epsilon}{2s} < \frac{\epsilon}{s}.
\]

Since \( S_b(Fx_k, Fx_k, Fx_k) = S_b(x_{k+1}, x_{k+1}, x_k) < \frac{\epsilon}{2s} \). Thus,

\[
S_b(x_k, x_k, Fz) \leq s[S_b(x_k, x_k, x_{k+1}) + S_b(x_k, x_k, x_{k+1}) + S_b(Fz, Fz, x_{k+1})]
\]

\[
= s[2S_b(x_k, x_k, x_{k+1}) + S_b(Fz, Fz, x_{k+1})]
\]

\[
\leq s[2\frac{\epsilon}{2s} + \frac{\epsilon}{s}] = \epsilon.
\]

Hence, \( F \) maps \( B(x_k, \epsilon) \) to itself. Since \( x_k \in B(x_k, \epsilon) \), we have \( Fx_k \in B(x_k, \epsilon) \). By repeating this process we get

\( F^m_{x_k} \in B(x_k, \epsilon) \) for all \( m \in \mathbb{N} \).

That is \( x_l \in B(x_k, \epsilon) \) for all \( l \geq k \). Hence

\[
S_b(x_m, x_m, x_l) < \epsilon \text{ for all } m, l > k.
\]

Therefore \( \{x_k\} \) is a Cauchy sequence and by the completeness of \( X \), there exists \( u \in X \) such that \( x_k \rightarrow u \) as \( k \rightarrow \infty \). Moreover, \( u = \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} x_k = F(u) \). Thus, \( F \) has \( u \) as a fixed point.

we prove now the uniqueness of the fixed point for \( F \). Since \( \alpha(t) = \psi^n(t) < t \) for any \( t > 0 \), let \( u \) and \( u_1 \) be two fixed points of \( F \).

\[
S_b(u, u, u_1) = S_b(Fu, Fu, Fu_1)
\]

\[
\leq \psi^n(u, u, u_1)
\]

\[
= \alpha(S_b(u, u, u_1))
\]

\[
\leq S_b(u, u, u_1),
\]

\[
\implies S_b(u, u, u_1) = 0 \implies u = u_1 \text{ and hence, } F \text{ has a unique fixed point in } X.
\]

On the other hand, \( T^{mk+r}(x) = F^k(T^r(x)) \rightarrow u \) as \( k \rightarrow \infty \). Hence, \( T^m x \rightarrow u \) as \( m \rightarrow \infty \) for every \( x \). That is \( u = \lim_{m \rightarrow \infty} T_{x_m} = T(u) \). Thereby, \( T \) has a fixed point.

The following results extend the results of [4] to the \( S_b \)-metric space.
Lemma 2.2. Let \((X, S_b)\) be a complete \(S_b\)-metric space. Then, for every descending sequence \(\{F_n\}_{n \geq 1}\) of nonempty closed subsets of \(X\) such that \(\text{diam}(F_n) \to 0\) as \(n \to \infty\). Therefore, the intersection \(\cap_{n=1}^{\infty} F_n\) contains one and only one point.

Proof. Let \(x_n\) be any point in \(F_n\). Because of the decrease of the sequence \(\{F_n\}_{n \geq 1}\), we have \(x_n, x_{n+1}, x_{n+2}, \ldots \in F_n\).

Given \(\epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(\text{diam}(F_{n_0}) < \epsilon\). We obtain \(x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots \in F_{n_0}\).

For \(m, n \geq n_0\), we have that

\[
S_b(x_n, x_n, x_m) \leq \text{diam}(F_{n_0}) < \epsilon.
\]

Hence, the sequence \(\{x_n\}_{n \geq 1}\) is a Cauchy sequence in the complete \(S_b\)-metric space. Thus, it is convergent. Let \(x \in X\) such that \(\lim_{n \to \infty} x_n = x\). Now, for any given \(n\) we have that \(x_n, x_{n+1}, x_{n+2}, \ldots \in F_n\). Therefore, \(x = \lim_{n \to \infty} x_n \in \overline{F_n} = F_n\) since \(F_n\) is closed. Thus, \(x \in \cap_{n=1}^{\infty} F_n\).

We now prove the uniqueness of \(x\). If \(y \in \cap_{n=1}^{\infty} F_n\) and \(y \neq x\), then \(S_b(x, x, y) = \alpha > 0\). There exists \(n \in \mathbb{N}\) large enough such that \(\text{diam}(F_n) < \alpha = S_b(x, x, y)\) which implies that \(y \neq \cap_{n=1}^{\infty} F_n\), which is a contradiction. \(\square\)

Definition 2.3. Let \((X, S_b)\) be a \(S_b\)-metric space, \(f : X \to \overline{\mathbb{R}}\) be a function.

- Let \(x_0 \in X\), \(f\) is a lower semi continuous at \(x_0\) if for every \(\epsilon > 0\) there exists a neighborhood \(U\) of \(x_0\) such that \(f(x) > f(x_0) - \epsilon\) for all \(x \in U\).

- \(f\) is said to be lower semi continuous if it is lower semi continuous at every point of \(X\).

Theorem 2.4. Let \((X, S_b)\) be a complete \(S_b\)-metric space (with \(s > 1\)), such that the \(S_b\)-metric is continuous and let \(f : X \to \overline{\mathbb{R}}\) be a a semi continuous function, proper and lower bounded mapping. Then for every \(x_0 \in X\) and \(\epsilon > 0\) with

\[
f(x_0) \leq \inf_{x \in X} f(x) + \epsilon,
\]

there exists a sequence \((x_n)_{n \in \mathbb{N}} \subset X\) and \(x_\epsilon \in X\) such that:

\[
i) \quad S_b(x_n, x_n, x_\epsilon) \leq \frac{\epsilon}{2^n}, \quad n \in \mathbb{N}, \tag{2.3}
\]

\[ii) \quad x_n \to x_\epsilon \text{ as } n \to \infty, \tag{2.4}
\]

\[iii) \quad f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x_\epsilon), \quad \text{for every } x \neq x_\epsilon, \tag{2.5}
\]

\[iv) \quad f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x_\epsilon) \leq f(x_0) \leq \inf_{x \in X} f(x) + \epsilon. \tag{2.6}
\]

Proof.

i) We consider the set

\[
Tx_0 = \{x \in X | f(x) + S_b(x, x, x_0) \leq f(x_0)\}. \tag{2.7}
\]

As \(f\) is a lower semi continuous mapping and \(x_0 \in Tx_0\), we obtain that \(Tx_0\) is nonempty and closed in \((X, S_b)\) and for every \(y \in Tx_0\)

\[
S_b(y, y, x_0) \leq f(x_0) - f(y) \leq f(x_0) - \inf_{x \in X} f(x) \leq \epsilon. \tag{2.8}
\]
We choose \( x_1 \in Tx_0 \) such that \( f(x_1) + S_b(x_1, x_1, x_0) \leq \inf_{x \in Tx_0} \{ f(x) + S_b(x, x, x_0) \} + \frac{\epsilon}{2s} \) and let

\[
Tx_1 = \{ x \in Tx_0 | f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i) \leq f(x_1) + S_b(x_1, x_1, x_0) \}. \tag{2.9}
\]

Inductively, we can suppose that \( x_{n-1} \in Tx_{n-2} \) was already chosen and we consider

\[
Tx_{n-1} := \{ x \in Tx_{n-2} | f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i) \leq f(x_{n-1}) + \sum_{i=0}^{n-2} \frac{1}{s^i} S_b(x_{n-1}, x_{n-1}, x_i) \}. \tag{2.10}
\]

Let \( x_n \in Tx_{n-1} \) such that

\[
f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i) \leq \inf_{x \in Tx_{n-1}} [ f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i) ] + \frac{\epsilon}{2^n s^n}. \tag{2.11}
\]

Define now the set

\[
Tx_n := \{ x \in Tx_{n-1} | f(x) + \sum_{i=0}^{n} \frac{1}{s^i} S_b(x, x, x_i) \leq f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i) \}. \tag{2.12}
\]

It is easy to see that the set \( Tx_n \) is nonempty and closed. Using the relations (2.11) and (2.12), we obtain for every \( y \in Tx_n \)

\[
f(y) + \sum_{i=0}^{n} \frac{1}{s^i} S_b(y, y, x_i) \leq f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i),
\]

which gives

\[
\frac{1}{s^n} S_b(y, y, x_n) \leq [ f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i) ] - [ f(y) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(y, y, x_i) ]
\]

\[
\leq [ f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i) ] - \inf_{x \in Tx_{n-1}} [ f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i) ]
\]

\[
\leq \frac{\epsilon}{2^n s^n}.
\]

Thus, for all \( y \in Tx_n \) we have

\[
S_b(y, y, x_n) \leq \frac{\epsilon}{2^n}. \tag{2.13}
\]

ii) From (2.13), we can deduce that \( S_b(y, y, x_n) \to 0 \) as \( n \to \infty \), so \( diam(Tx_n) \to 0 \). As \( (X, S_b) \) is a complete \( S_b \)-metric space and from Lemma 2.2 we have \( \cap_{n=0}^{\infty} Tx_n = \{ x_\epsilon \} \). Using the equations (2.8) and (2.13) we obtain that \( x_\epsilon \in X \) satisfies (2.3). Therefore,

\[
x_n \to x_\epsilon \text{ as } n \to \infty.
\]

iii) As \( x_\epsilon \) is the single intersection of all the sets \( Tx_n \), so for all \( x \neq x_\epsilon \), we have \( x \notin \cap_{n=0}^{\infty} Tx_n \). Thus, there exists \( m \in \mathbb{N} \) such that

\[
x \in Tx_{m-1} \text{ and } x \notin Tx_m. \tag{2.14}
\]
Using (2.12) and (2.14), we obtain
\[ f(x) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x, x, x_i) > f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i). \]  
(2.15)

Thereby, (2.5) holds.

iv) Using (2.14) and the definition of the set \( T_{x_{n-1}} \), we obtain
\[ f(x) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i) \leq f(x_{m-1}) + \sum_{i=0}^{m-2} \frac{1}{s^i} S_b(x_{m-1}, x_{m-1}, x_i). \]  
(2.16)

Similarly, by applying (2.16) to \( x_{m-1} \) we have that
\[ f(x_{m-1}) + \sum_{i=0}^{m-2} \frac{1}{s^i} S_b(x_{m-1}, x_{m-1}, x_i) \leq f(x_{m-2}) + \sum_{i=0}^{m-3} \frac{1}{s^i} S_b(x_{m-2}, x_{m-2}, x_i). \]  
(2.17)

By repeating this procedure enough times, we obtain
\[ f(x_0) \geq f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i). \]

Moreover, for every \( q \geq m \), we have
\[ f(x_0) \geq f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i) \geq f(x_q) + \sum_{i=0}^{q-1} \frac{1}{s^i} S_b(x_q, x_q, x_i) \geq f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} S_b(x_\epsilon, x_\epsilon, x_i). \]

Then, (2.6) holds.

Next, we state this immediate consequence.

**Corollary 2.5.** Let \( (X, S_b) \) be a complete \( S_b \)-metric space (with \( s > 1 \)), such that the \( S_b \)-metric is continuous and let \( f : X \to \mathbb{R} \) be a lower semi continuous, proper and lower bounded mapping. Then for every \( \epsilon > 0 \) there exists a sequence \( (x_n)_{n \in \mathbb{N}} \subset X \) and \( x^* \in X \) such that:

i) \( x_n \to x_\epsilon \), as \( n \to \infty \) \( x_\epsilon \in X \),  
(2.18)

ii) \( f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \leq \inf_{x \in X} f(x) + \epsilon \),  
(2.19)

iii) \( f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \) for any \( x \in X \).  
(2.20)

**Theorem 2.6.** Let \( (X, S_b) \) be a complete \( S_b \)-metric space (with \( s > 1 \)), such that the \( S_b \)-metric is continuous and let \( T : X \to X \) be an operator for which there exists a lower semi continuous mapping \( f : X \to \mathbb{R} \), such that:

i) \( S_b(u, u, v) + sS_b(u, u, Tu) \geq S_b(Tu, Tu, v) \),  
(2.21)

ii) \( \frac{s^2}{s-1} S_b(u, u, Tu) \leq f(u) - f(Tu) \), for any \( u, v \in X \).  
(2.22)

Then \( T \) has at least one fixed point.
Proof. Assume that for all $x \in X$ we have that $Tx \neq x$. Using Corollary 2.5 for $f$, we obtain that, for each $\epsilon > 0$ there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that $x_n \to x_\epsilon$, as $n \to \infty$, $x_\epsilon \in X$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n)$$

for any $x \in X$.

Since the above inequality holds for every $x \in X$, let put $x := Tx_\epsilon$ and since $Tx_\epsilon \neq x_\epsilon$, we get that

$$f(x_\epsilon) - f(Tx_\epsilon) < \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(Tx_\epsilon, Tx_\epsilon, x_n) - \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n).$$

(2.23)

Let $u = x_\epsilon$ and $v = x_n$ in (2.21), we obtain

$$S_b(x_\epsilon, x_\epsilon, x_n) + sS_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \geq S_b(Tx_\epsilon, Tx_\epsilon, x_n).$$

(2.24)

From (2.23) and (2.24) we have

$$f(x_\epsilon) - f(Tx_\epsilon) < \sum_{n=0}^{\infty} \frac{s}{s^n} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon)$$

$$\leq sS_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \sum_{n=0}^{\infty} \frac{1}{s^n}$$

$$\leq \frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon).$$

(2.25)

In (2.22) we choose $u = x_\epsilon$. Then

$$\frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon).$$

(2.26)

From the inequalities (2.25) and (2.26) we get that

$$\frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon) < \frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon),$$

which is a absurd. Therefore, there exists $x^* \in X$ such that $x^* \in Tx^*$.

\hfill \Box

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University, Saudi Arabia, for funding this research work.

References


